

# Multi-Bernoulli filter for superpositional sensors

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**Abstract**—The superpositional sensor model encompasses an important class of sensors such as acoustic sensors and radio-frequency sensors used for multi-target tracking. Recently, random finite set based moment filters such as PHD and CPHD filters have been developed for superpositional sensors. In this paper we derive multi-Bernoulli filter equations for superpositional sensors. The multi-Bernoulli update is derived by defining a conditional PHD for each component of the multi-Bernoulli random finite set and then following an approach similar to that used in deriving the CPHD filter update equation for superpositional sensors. The cardinality distribution is also updated along with the conditional PHD.

**Keywords:** multi-Bernoulli filter, random finite set, superpositional sensors, PHD filter, CPHD filter, multi-target tracking.

## I. INTRODUCTION

Multi-target tracking is the problem of detecting and tracking a possibly time varying number of mobile targets. The random vector based formulation of this problem is inefficient for implementation because of the time varying number of targets. To address this issue, Mahler [1], [2] proposed the random finite set (RFS) based formulation of the multi-target tracking problem and developed the tools of finite set statistics (FISST). This framework allows us to efficiently model the problem of multi-target multi-source target tracking and derive new tracking algorithms.

The commonly studied sensors in most of the literature concerning random finite set tracking have the following modelling assumptions: (i) each target causes either one or no measurement; and (ii) each measurement is either caused by a single target or clutter. We are interested in the multi-target tracking problem when *superpositional sensors* are used to make observations. The observations made by superpositional sensors are functions of all the targets present rather than just one of them. These sensors have the following modelling assumptions: (i) each target can contribute to any number of measurements; (ii) each measurement is potentially affected by multiple targets in an additive fashion; and (iii) measurements are not independent. Many sensors belong to the category of superpositional sensors. Examples include direction-of-arrival sensors for linear antenna arrays [3], antenna arrays in multi-user detection for wireless communication networks [4], acoustic amplitude sensors [5], and radio frequency (RF) tomographic tracking systems [6].

Probability hypothesis density (PHD) and cardinalized probability hypothesis density (CPHD) filters are first-moment

based filters developed using the RFS approach and the tools of FISST. These filters were derived by Mahler in [7] and [8] for the case of standard sensors. The PHD filter for superpositional sensors was derived by Thouin, Nannuru and Coates in [9] where it was called the additive likelihood moment (ALM) filter<sup>1</sup>. In [11] Mahler and El-Fallah developed the CPHD filter for superpositional sensors. Nannuru et al. provided a summary of the PHD and CPHD filters for superpositional sensors in [12] and presented auxiliary particle filter algorithms. The paper described numerical simulations of multi-target tracking in environments of acoustic sensor network and radio frequency sensor network.

The PHD and CPHD filters assume the underlying distributions to be Poisson and i.i.d. cluster multi-target random process respectively. The multi-Bernoulli filter is based on a multi-Bernoulli modelling of the underlying RFS. A multi-Bernoulli RFS is a union of finitely many Bernoulli random finite sets, each of which models the probability of existence and location distribution of a single target. The multi-Bernoulli filter was proposed by Mahler in [2] and approximately propagates the full multi-target distribution for standard sensors. Vo et al. in [13] present a modified version of the filter which corrects for the cardinality bias of the original formulation.

The corresponding Bernoulli filter was used for single target tracking by authors in [14], [15]. The multi-Bernoulli filter was adapted for estimation and detection of multiple objects from image observations in track-before-detect applications [16] under the assumption that the likelihood has a separable form. This assumption is valid when the objects are non-overlapping. Hoseinnezhad et al. [17] used this filter for tracking multiple targets in background subtracted image sequences. Williams [18] has proposed a hybrid combination of the PHD and multi-Bernoulli filter for multi-target tracking resulting in fast track initiation and fewer Bernoulli components. The particle based implementations of the multi-Bernoulli filter have been discussed in [13], [16]. Lian et al. [19] have performed convergence analysis of the sequential Monte Carlo (SMC) implementations of the multi-Bernoulli filters.

In this paper we derive the approximate multi-Bernoulli filter for superpositional sensors. Our approach is to define a *conditional PHD* for each Bernoulli component of the multi-

<sup>1</sup>An error in the main update equation of ALM filter in [9] was corrected in an errata [10]; the correct equations were also presented in [11].

Bernoulli RFS. The conditioning is considered with respect to the event that if the element  $\mathbf{x}$  is present, it is generated by the Bernoulli RFS component under consideration. This allows us to update the PHD for each individual Bernoulli RFS component and hence propagate the multi-Bernoulli filter parameters. We also update the cardinality distribution of the multi-Bernoulli RFS in order to propagate the probabilities of existence for each component. The techniques applied in the derivation of PHD and CPHD filters for superpositional sensors [12] are employed to obtain approximate but computationally tractable multi-Bernoulli filter update equations.

The paper is organized as follows. In Section II we summarize the problem of multi-target tracking using superpositional sensors. Section III provides a brief overview of the multi-Bernoulli RFS and its statistical properties which are later used for deriving the filter equations. The multi-Bernoulli filter prediction step is described in Section IV-A. The conditional PHD update equations for superpositional sensors are derived in Section IV-B. The cardinality update is derived in Section IV-D. We summarize the contribution of the paper and make concluding remarks in Section V.

## II. PROBLEM FORMULATION

In this section we describe the problem of multi-target tracking when the observations are of superpositional form. The true system state at time  $k$  is a finite set  $x_k$  which can be modeled as a realization of a random finite set  $X_k = \{\mathbf{x}_{k,1}, \dots, \mathbf{x}_{k,n_k}\}$  where  $n_k \geq 0$  is number of targets present at time  $k$ . The single target state dimension is  $n_{\mathbf{x}}$ , hence  $\mathbf{x}_{k,i} \in R^{n_{\mathbf{x}}} \forall i$ . We assume that the individual target dynamics are specified according to a Markovian model of the form  $\mathbf{x}_{k+1,i} = f_{k+1|k}(\mathbf{x}_{k,i}, \mathbf{u}_k)$  where  $\mathbf{u}_k$  is the noise.

The true state is hidden but can be inferred from the measurements  $\mathbf{z}_k = [z_k^1 \dots z_k^{n_z}]$ . The collection of measurements up to time  $k$  is denoted by  $Z^{[k]} = \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ . In the superpositional sensor model, the likelihood function relating the observation  $\mathbf{z}_k$  and true state  $x_k$  at time  $k$  is of the form:

$$\begin{aligned} h_{\mathbf{z}_k}(x_k) &= h_{\mathbf{z}_k}(r(x_k)) \\ &= h_{\mathbf{z}_k} \left( \sum_{\mathbf{x} \in x_k} g(\mathbf{x}) \right) \end{aligned} \quad (1)$$

where  $h_{\mathbf{z}_k}$  is a real-valued function and  $g$  and  $r$  are (potentially non-linear) functions mapping to vectors of reals. The function  $r$  operates on a finite set whereas the function  $g$  operates on the target states that are members of the set. When the sensor observation noise is Gaussian with zero mean and covariance matrix  $\Sigma_r$ , the likelihood takes the form,

$$h_{\mathbf{z}_k}(x_k) = \mathcal{N}_{\Sigma_r}(\mathbf{z}_k - r(x_k)) \quad (2)$$

where  $\mathcal{N}_{\Sigma}(\mathbf{z})$  denotes the Gaussian density function with zero mean and covariance matrix  $\Sigma$  evaluated at  $\mathbf{z}$ .

The multi-target tracking problem is to estimate the number and location of all the targets present at every time step  $k$ . This is obtained from the posterior distribution of the multi-target state at each time step  $k$  given all the observations up to

time  $k$ , i.e.,  $f(X_k|Z^{[k]})$ . The optimal solution is to propagate the posterior distribution using the recursive Bayes equations within the FISST framework. In this paper we solve the Bayes equations in an approximate manner under the assumption that the multi-target state is a multi-Bernoulli RFS at each step.

## III. MULTI-BERNOULLI RANDOM FINITE SETS

In this section we provide an overview of multi-Bernoulli random finite sets and some of their statistical properties. A multi-Bernoulli RFS is union of finitely many Bernoulli random finite sets. Hence we start with a brief review of Bernoulli random finite sets.

A Bernoulli random finite set is an empty set with probability  $1 - r$  or is a singleton set with probability  $r$  with its element  $\mathbf{x}$  distributed according to the probability density  $p(\mathbf{x})$ . It is completely described by specifying the parameter set  $\{r, p(\mathbf{x})\}$ . The multi-target probability density of a Bernoulli RFS is given by

$$f(X) = \begin{cases} 1 - r, & \text{if } X = \phi \\ r \cdot p(\mathbf{x}), & \text{if } X = \{\mathbf{x}\} \\ 0 & \text{if } |X| > 1. \end{cases} \quad (3)$$

where  $|X|$  denotes the cardinality of the RFS  $X$ . The cardinality distribution  $\pi(n)$  of the Bernoulli RFS can be easily seen to be given by

$$\pi(n) = \int_{|X|=n} f(X) \delta X \quad (4)$$

$$\pi(n) = \begin{cases} 1 - r & \text{if } n = 0 \\ r & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases} \quad (5)$$

The first-moment or the probability hypothesis density (PHD) of the Bernoulli RFS is given by

$$\begin{aligned} D(\mathbf{x}) &= \int f(\{\mathbf{x}\} \cup W) \delta W \\ &= r \cdot p(\mathbf{x}). \end{aligned} \quad (6)$$

Note that the integrals above are set integrals [2] and are defined as

$$\int f(X) \delta X = f(\phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f(\{\mathbf{x}_1, \dots, \mathbf{x}_n\}) d\mathbf{x}_1 \dots d\mathbf{x}_n$$

A multi-Bernoulli RFS  $\chi$  having  $M$  components is a union of a finite number of Bernoulli random finite sets and can be written as

$$\chi = \chi_1 \cup \chi_2 \cup \dots \cup \chi_M \quad (8)$$

where each of the  $\chi_i$  is a Bernoulli RFS. Let the parameters of the  $i^{\text{th}}$  Bernoulli random finite set be given by  $\{r_i, p_i(\mathbf{x})\}$ . The parameter set of the corresponding multi-Bernoulli RFS is denoted by  $\{r_i, p_i(\mathbf{x})\}_{i=1}^M$ . The multi-target probability density of a multi-Bernoulli RFS [13] is:

$$f(\{\mathbf{x}_1, \mathbf{x}_2 \dots \mathbf{x}_n\}) = Q_0 \times \sum_{1 \leq i_1 \neq \dots \neq i_n \leq M} \prod_{j=1}^n \frac{r_{i_j} p_{i_j}(\mathbf{x}_j)}{1 - r_{i_j}} \quad (9)$$

where  $Q_0 = \prod_{j=1}^M (1 - r_j)$  and  $n \leq M$ . The cardinality distribution of the multi-Bernoulli RFS is given by

$$\pi(n) = \begin{cases} Q_0 & \text{if } n = 0 \\ \frac{Q_0}{n!} \times \sum_{1 \leq i_1 \neq \dots \neq i_n \leq M} \prod_{j=1}^n \frac{r_{i_j}}{1 - r_{i_j}} & \text{if } n \leq M \\ 0 & \text{if } n > M. \end{cases} \quad (10)$$

The PHD of the multi-Bernoulli RFS (Ex. 91, Chap. 16, [2]) can be shown to be

$$D(\mathbf{x}) = \sum_{i=1}^M r_i \cdot p_i(\mathbf{x}). \quad (11)$$

Similarly the second factorial moment density of the multi-Bernoulli RFS can be computed to be

$$D(\{\mathbf{x}_1, \mathbf{x}_2\}) = \int f(\{\mathbf{x}_1, \mathbf{x}_2\} \cup W) \delta W \quad (12)$$

$$= \sum_{i=1}^M \sum_{j=1, j \neq i}^M r_i \cdot r_j \cdot p_i(\mathbf{x}_1) \cdot p_j(\mathbf{x}_2) \quad (13)$$

$$= D(\mathbf{x}_1)D(\mathbf{x}_2) - \sum_{i=1}^M r_i^2 \cdot p_i(\mathbf{x}_1) \cdot p_i(\mathbf{x}_2). \quad (14)$$

We now define the conditional PHD,  $D^i(\mathbf{x})$ , corresponding to the  $i^{\text{th}}$  component of the multi-Bernoulli RFS. It is obtained by conditioning on the following event: if  $\mathbf{x}$  is a member of the multi-target state, then element  $\mathbf{x}$  is generated by the Bernoulli random finite set  $\chi_i$ . We denote this event by  $(\mathbf{x} \leftarrow i)$ , and define the conditional PHD as

$$D^i(\mathbf{x}) = \int f(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow i) \delta W \quad (15)$$

Expanding using the definition of set integral we have

$$D^i(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int f(\{\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n\} | \mathbf{x} \leftarrow i) d\mathbf{y}_1 \cdots d\mathbf{y}_n \quad (16)$$

$$= r_i \cdot p_i(\mathbf{x}) \times \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbf{x} \notin \{\mathbf{y}_1, \dots, \mathbf{y}_n\}} f^{\bar{i}}(\{\mathbf{y}_1, \dots, \mathbf{y}_n\}) d\mathbf{y}_1 \cdots d\mathbf{y}_n \quad (17)$$

$$= r_i \cdot p_i(\mathbf{x}) \times \int_{\mathbf{x} \notin W} f^{\bar{i}}(W) \delta W \quad (18)$$

$$= r_i \cdot p_i(\mathbf{x}) \quad (19)$$

where  $f^{\bar{i}}(Y)$  is the multi-target probability density of the multi-Bernoulli RFS  $\chi$  in (8) but excluding the  $i^{\text{th}}$  Bernoulli component  $\chi_i$ . Equation (18) follows from equation (17) because the probability of the event  $\mathbf{x} \in W$  is zero. Defining the conditional PHD  $D^i(\mathbf{x})$  is a key step in deriving the update equation for the multi-Bernoulli filter for superpositional sensors.

#### IV. MULTI-BERNOULLI FILTER FOR SUPERPOSITIONAL SENSORS

The multi-Bernoulli filter propagates over time the parameters of the multi-Bernoulli RFS representing the multi-target state. The propagation is done in two stages, prediction and update. The prediction step propagates the components using the known motion model. The motion model accounts for the survival of targets from the previous time step and also for the birth of new targets. Target spawning is not considered in this paper. The update step re-evaluates the component parameters using the most recent observation. Developing the update step for the multi-Bernoulli filter under the superpositional sensor model assumption is the major contribution of this paper. It consists of two parts: the conditional PHD update and the cardinality distribution update.

##### A. Multi-Bernoulli filter prediction

The superpositional observation model does not change the multi-Bernoulli prediction equations [2], [16]. Let  $\{r_{k,i}, p_{k,i}(\mathbf{x})\}_{i=1}^{M_k}$  be the parameter set of the posterior multi-Bernoulli density at time  $k$ . Let the predicted multi-Bernoulli set have  $M_{k+1|k}$  elements. For brevity we use the following abbreviated notation for predicted multi-Bernoulli parameters:

$$r_i = r_{k+1|k,i} \quad (20)$$

$$p_i(\mathbf{x}) = p_{k+1|k,i}(\mathbf{x}) \quad (21)$$

The predicted multi-Bernoulli RFS parameters are

$$\{r_i, p_i(\mathbf{x})\}_{i=1}^{M_{k+1|k}} = \{r_{P,i}, p_{P,i}(\mathbf{x})\}_{i=1}^{M_k} \cup \{r_{B,i}, p_{B,i}(\mathbf{x})\}_{i=M_k+1}^{M_{k+1|k}} \quad (22)$$

where  $\{r_{P,i}, p_{P,i}(\mathbf{x})\}_{i=1}^{M_k}$  are the parameters of targets propagated from the previous time step and  $\{r_{B,i}, p_{B,i}(\mathbf{x})\}_{i=M_k+1}^{M_{k+1|k}}$  are the parameters of newly born targets. The predicted target parameters at time  $k+1$  are related to the posterior parameters at time  $k$  as

$$r_{P,i} = r_{k,i} \times \langle p_{k,i}, p_s \rangle \quad (23)$$

$$p_{P,i}(\mathbf{x}) = \frac{\langle f_{k+1|k}(\mathbf{x}|\cdot), p_{k,i} p_s \rangle}{\langle p_{k,i}, p_s \rangle} \quad (24)$$

where the scalar product is defined as  $\langle a, b \rangle = \int a(\mathbf{x})b(\mathbf{x})d\mathbf{x}$  and  $p_s(\mathbf{x})$  is the survival probability. The parameters of newly born targets are given by the target birth model.

##### B. Multi-Bernoulli filter conditional PHD update

We now derive the multi-Bernoulli filter conditional PHD update equation for superpositional sensors. For the purpose of presenting a simpler and clearer derivation, we assume that no new targets are added in the update step and hence  $M_{k+1} = M_{k+1|k}$ . Generalization of the equations is straightforward. Let the updated multi-Bernoulli parameters be abbreviated as follows:

$$r'_i = r_{k+1,i} \quad (25)$$

$$p'_i(\mathbf{x}) = p_{k+1,i}(\mathbf{x}) \quad (26)$$

Thus the parameters of the posterior multi-Bernoulli RFS are  $\{r'_i, p'_i(\mathbf{x})\}_{i=1}^{M_{k+1}}$ . We now turn our attention to the predicted and posterior conditional PHD at time  $k+1$ ,  $D_{k+1|k}^i(\mathbf{x})$  and  $D_{k+1}^i(\mathbf{x})$ , conditioned on the following event: if element  $\mathbf{x}$  is a member of the multi-target state, then  $\mathbf{x}$  is generated by Bernoulli RFS  $\chi_i$ . These PHDs are given from equation (19) as

$$D_{k+1|k}^i(\mathbf{x}) = r_i \cdot p_i(\mathbf{x}) \quad (27)$$

$$D_{k+1}^i(\mathbf{x}) = r'_i \cdot p'_i(\mathbf{x}) \quad (28)$$

By definition we also have

$$D_{k+1}^i(\mathbf{x}) = \int f_{k+1|k+1}(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow i) \delta W \quad (29)$$

Using equation (28) and applying Bayes' rule to equation (29) we get

$$\begin{aligned} r'_i \cdot p'_i(\mathbf{x}) &= \int f_{k+1|k+1}(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow i) \delta W \quad (30) \\ &= \frac{\int f_{k+1}(\mathbf{z}_{k+1} | \{\mathbf{x}\} \cup W, \mathbf{x} \leftarrow i) f_{k+1|k}(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow i) \delta W}{f_{k+1}(\mathbf{z}_{k+1} | Z^{[k]}, \mathbf{x} \leftarrow i)} \\ &= r_i \cdot p_i(\mathbf{x}) \frac{\int f_{k+1}(\mathbf{z}_{k+1} | \{\mathbf{x}\} \cup W) \times f_{k+1|k}^{\bar{i}}(W) \delta W}{f_{k+1}(\mathbf{z}_{k+1} | Z^{[k]}, \mathbf{x} \leftarrow i)} \quad (31) \end{aligned}$$

where  $f_{k+1|k}^{\bar{i}}(W)$  is the multi-target distribution defined as

$$f_{k+1|k}^{\bar{i}}(W) = \frac{f_{k+1|k}(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow i)}{r_i \cdot p_i(\mathbf{x})} \quad (32)$$

This is a valid distribution which integrates to 1 using the results in equations (15)-(19). For any  $W$  such that  $\mathbf{x} \notin W$  it corresponds to the predicted multi-target distribution excluding the contribution from the  $i^{\text{th}}$  Bernoulli component  $\chi_i$ . Thus we have

$$\begin{aligned} r'_i \cdot p'_i(\mathbf{x}) &= r_i \cdot p_i(\mathbf{x}) \\ &\times \frac{\int f_{k+1}(\mathbf{z}_{k+1} | \{\mathbf{x}\} \cup W) \times f_{k+1|k}^{\bar{i}}(W) \delta W}{\int f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W | Z^{[k]}, \mathbf{x} \leftarrow i) \delta W} \end{aligned}$$

To simplify the denominator, we note that

$$\begin{aligned} &\int f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W | Z^{[k]}, \mathbf{x} \leftarrow i) \delta W \\ &= \int f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W) \delta W \quad (33) \end{aligned}$$

This is because the conditional event  $(\mathbf{x} \leftarrow i)$  has no effect on the integral. To see this consider the following decomposition of the integral on the left:

$$\begin{aligned} &\int f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W | Z^{[k]}, \mathbf{x} \leftarrow i) \delta W \\ &= \int_{\mathbf{x} \in W} f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W | Z^{[k]}, \mathbf{x} \leftarrow i) \delta W \\ &\quad + \int_{\mathbf{x} \notin W} f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W | Z^{[k]}) \delta W \quad (34) \end{aligned}$$

But the first integral is zero since  $\mathbf{x} \in W$  is a zero probability event. To the second integral we can add the following term,

which has zero probability and thus does not affect the evaluation of the integral:

$$\int_{\mathbf{x} \in W} f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W | Z^{[k]}) \delta W \quad (35)$$

This leads to the expression on the right hand side of equation (33). Hence we have

$$\begin{aligned} r'_i \cdot p'_i(\mathbf{x}) &= r_i \cdot p_i(\mathbf{x}) \\ &\times \frac{\int f_{k+1}(\mathbf{z}_{k+1} | \{\mathbf{x}\} \cup W^{\bar{i}}) \times f_{k+1|k}^{\bar{i}}(W^{\bar{i}}) \delta W^{\bar{i}}}{\int f_{k+1}(\mathbf{z}_{k+1} | W) \times f_{k+1|k}(W) \delta W} \end{aligned}$$

In above notation  $W$  and  $W^{\bar{i}}$  correspond to the random finite sets with distributions  $f_{k+1|k}(W)$  and  $f_{k+1|k}^{\bar{i}}(W^{\bar{i}})$ , respectively.

### C. Computationally tractable approximations for the case of Gaussian noise

Using the Gaussian sensor noise assumption and the superpositional likelihood model, we have:

$$\begin{aligned} r'_i \cdot p'_i(\mathbf{x}) &= r_i \cdot p_i(\mathbf{x}) \\ &\times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - g(\mathbf{x}) - r(W^{\bar{i}})) \times f_{k+1|k}^{\bar{i}}(W^{\bar{i}}) \delta W^{\bar{i}}}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - r(W)) \times f_{k+1|k}(W) \delta W} \end{aligned}$$

Applying the transformation  $\mathbf{y}^{\bar{i}} = r(W^{\bar{i}})$  in the numerator and  $\mathbf{y} = r(W)$  in the denominator and using the formula for transformation of variables we have,

$$\begin{aligned} r'_i \cdot p'_i(\mathbf{x}) &= r_i \cdot p_i(\mathbf{x}) \\ &\times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mathbf{y}^{\bar{i}}) \times Q_{k+1|k}^{\bar{i}}(\mathbf{y}^{\bar{i}}) d\mathbf{y}^{\bar{i}}}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - \mathbf{y}) \times Q_{k+1|k}(\mathbf{y}) d\mathbf{y}} \quad (36) \end{aligned}$$

where  $Q_{k+1|k}(\mathbf{y})$  and  $Q_{k+1|k}^{\bar{i}}(\mathbf{y}^{\bar{i}})$  are the probability distributions of the random vectors  $\mathbf{y}$  and  $\mathbf{y}^{\bar{i}}$  respectively. Using the approximations  $Q_{k+1|k}(\mathbf{y}) \approx \mathcal{N}_{C_{k+1}}(\mathbf{y} - \mu_{k+1})$  and  $Q_{k+1|k}^{\bar{i}}(\mathbf{y}^{\bar{i}}) \approx \mathcal{N}_{C_{k+1}^{\bar{i}}}(\mathbf{y}^{\bar{i}} - \mu_{k+1}^{\bar{i}})$ ,

$$\begin{aligned} r'_i \cdot p'_i(\mathbf{x}) &\approx r_i \cdot p_i(\mathbf{x}) \\ &\times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mathbf{y}^{\bar{i}}) \times \mathcal{N}_{C_{k+1}^{\bar{i}}}(\mathbf{y}^{\bar{i}} - \mu_{k+1}^{\bar{i}}) d\mathbf{y}^{\bar{i}}}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - \mathbf{y}) \times \mathcal{N}_{C_{k+1}}(\mathbf{y} - \mu_{k+1}) d\mathbf{y}} \\ r'_i \cdot p'_i(\mathbf{x}) &\approx r_i \cdot p_i(\mathbf{x}) \cdot \frac{\mathcal{N}_{\Sigma_r + C_{k+1}^{\bar{i}}}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mu_{k+1}^{\bar{i}})}{\mathcal{N}_{\Sigma_r + C_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})} \quad (37) \end{aligned}$$

where  $\mu_{k+1}$  and  $C_{k+1}$  are the mean and covariance matrix of the distribution  $Q_{k+1|k}(\mathbf{y})$  and  $\mu_{k+1}^{\bar{i}}$  and  $C_{k+1}^{\bar{i}}$  are the mean and covariance matrix of the distribution  $Q_{k+1|k}^{\bar{i}}(\mathbf{y}^{\bar{i}})$ . These mean and covariance matrix parameters can be found using the quadratic version of Campbell's theorem [11], [12] and are given by

$$\mu_{k+1} = \sum_{i=1}^{M_{k+1|k}} r_i \cdot s_i \quad (38)$$

$$C_{k+1} = \sum_{i=1}^{M_{k+1|k}} (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) \quad (39)$$

where  $s_i = \langle p_i, g \rangle$ ,  $v_i = \langle p_i, gg^T \rangle$  and

$$\mu_{k+1}^{\bar{i}} = \mu_{k+1} - r_i \cdot s_i \quad (40)$$

$$C_{k+1}^{\bar{i}} = C_{k+1} - (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) \quad (41)$$

The derivation of these parameters is provided in Appendix A.

#### D. Multi-Bernoulli filter cardinality update

The conditional PHD update equation propagates the product of the form  $r_i \cdot p_i(\mathbf{x})$  for each Bernoulli component. Since the joint product is propagated, it is not possible to infer the update equation for the probability of existence. Hence we additionally propagate the cardinality distribution of the multi-Bernoulli RFS to capture the information in existence probabilities. The cardinality distribution at time  $k+1$  is given by

$$\pi_{k+1}(n) = \int_{|W|=n} f_{k+1|k+1}(W) \delta W \quad (42)$$

$$= \frac{\int_{|W|=n} f_{k+1}(\mathbf{z}_{k+1}|W) f_{k+1|k}(W) \delta W}{f_{k+1}(\mathbf{z}_{k+1}|Z^{[k]})} \quad (43)$$

$$= \frac{\int_{|W|=n} f_{k+1}(\mathbf{z}_{k+1}|W) f_{k+1|k}(W) \delta W}{\int f_{k+1}(\mathbf{z}_{k+1}|W) f_{k+1|k}(W) \delta W} \quad (44)$$

$$= \pi_{k+1|k}(n) \frac{\int f_{k+1}(\mathbf{z}_{k+1}|W) f_{k+1|k}^n(W) \delta W}{\int f_{k+1}(\mathbf{z}_{k+1}|W) f_{k+1|k}(W) \delta W} \quad (45)$$

where,

$$f_{k+1|k}^n(W) = \frac{1}{\pi_{k+1|k}(n)} \cdot \delta_{|W|=n} \cdot f_{k+1|k}(W) \quad (46)$$

Using the Gaussian sensor noise assumption and transformation of variables as before, we have the expression

$$\begin{aligned} \pi_{k+1}(n) &\approx \pi_{k+1|k}(n) \\ &\times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - \mathbf{y}^n) \times \mathcal{N}_{C_{k+1}^n}(\mathbf{y}^n - \mu_{k+1}^n) d\mathbf{y}^n}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - \mathbf{y}) \times \mathcal{N}_{C_{k+1}}(\mathbf{y} - \mu_{k+1}) d\mathbf{y}} \\ \pi_{k+1}(n) &\approx \pi_{k+1|k}(n) \cdot \frac{\mathcal{N}_{\Sigma_r + C_{k+1}^n}(\mathbf{z}_{k+1} - \mu_{k+1}^n)}{\mathcal{N}_{\Sigma_r + C_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})} \quad (47) \end{aligned}$$

where  $\mu_{k+1}$  and  $C_{k+1}$  are given from equations (38) and (39). The quantities  $\mu_{k+1}^n$  and  $C_{k+1}^n$  are obtained using the quadratic form of Campbell's theorem and the PHD of the multi-target distribution  $f_{k+1|k}^n(W)$  and are given by

$$\mu_{k+1}^n = \frac{1}{\pi_{k+1|k}(n)} \sum_{i=1}^{M_{k+1|k}} r_i \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \cdot s_i \quad (48)$$

$$\begin{aligned} C_{k+1}^n &= \frac{1}{\pi_{k+1|k}(n)} \left( \sum_{i=1}^{M_{k+1|k}} r_i \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \cdot v_i \right. \\ &\quad \left. + \sum_{i \neq j} r_i \cdot r_j \cdot \pi_{k+1|k}^{\bar{i}, \bar{j}}(n-2) \cdot s_i s_j^T \right) - \mu_{k+1}^n (\mu_{k+1}^n)^T \quad (49) \end{aligned}$$

The above parameters are derived in Appendix B.

## V. CONCLUSIONS

In this paper we developed a computationally tractable approximate multi-Bernoulli filter for superpositional sensors. The key step in deriving the filter update equation is defining a conditional PHD. Updating this conditional PHD for each component of the multi-Bernoulli RFS allows us to propagate the multi-target posterior distribution. The probabilities of existence of individual components are indirectly propagated by approximately updating the cardinality distribution. The proposed filter equations can be implemented using SMC methods, with the conditional PHD being propagated by using one particle filter per Bernoulli component. In future work, we will describe particle filter based implementations of the multi-Bernoulli filter and assess performance in comparison to the PHD and CPHD filters developed previously.

### APPENDIX A

Using the quadratic version of Campbell's theorem we have

$$\mu_{k+1} = \int g(\mathbf{x}) \cdot D_{k+1|k}(\mathbf{x}) d\mathbf{x}$$

$$\begin{aligned} C_{k+1} &= \int g(\mathbf{x}) \cdot g(\mathbf{x})^T \cdot D_{k+1|k}(\mathbf{x}) d\mathbf{x} \\ &\quad + \int \int g(\mathbf{x}_1) \cdot g(\mathbf{x}_2)^T \left( D_{k+1|k}(\{\mathbf{x}_1, \mathbf{x}_2\}) \right. \\ &\quad \left. - D_{k+1|k}(\mathbf{x}_1) \cdot D_{k+1|k}(\mathbf{x}_2) \right) d\mathbf{x}_1 d\mathbf{x}_2 \end{aligned}$$

For the multi-Bernoulli distribution, using the expression for the PHD and the second-moment density from equations (11) and (14) we have

$$\mu_{k+1} = \int g(\mathbf{x}) \cdot \left( \sum_{i=1}^{M_{k+1|k}} r_i \cdot p_i(\mathbf{x}) \right) d\mathbf{x} \quad (50)$$

$$= \sum_{i=1}^{M_{k+1|k}} r_i \times \int g(\mathbf{x}) \cdot p_i(\mathbf{x}) d\mathbf{x} \quad (51)$$

$$= \sum_{i=1}^{M_{k+1|k}} r_i \cdot s_i, \quad \text{where } s_i = \langle p_i, g \rangle \quad (52)$$

$$\begin{aligned} C_{k+1} &= \int g(\mathbf{x}) g(\mathbf{x})^T \left( \sum_{i=1}^{M_{k+1|k}} r_i \cdot p_i(\mathbf{x}) \right) d\mathbf{x} \\ &\quad - \int \int g(\mathbf{x}_1) g(\mathbf{x}_2)^T \left( \sum_{i=1}^{M_{k+1|k}} r_i^2 \cdot p_i(\mathbf{x}_1) \cdot p_i(\mathbf{x}_2) \right) d\mathbf{x}_1 d\mathbf{x}_2 \\ &= \sum_{i=1}^{M_{k+1|k}} (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T), \quad \text{where } v_i = \langle p_i, gg^T \rangle \quad (53) \end{aligned}$$

The parameters  $\mu_{k+1}^{\bar{i}}$  and  $C_{k+1}^{\bar{i}}$  can be obtained in a similar manner by considering the predicted multi-Bernoulli distribution but excluding the  $i^{\text{th}}$  Bernoulli component. Hence we

have

$$\mu_{k+1}^{\bar{i}} = \sum_{j=1, j \neq i}^{M_{k+1|k}} r_j \cdot s_j = \mu_{k+1} - r_i \cdot s_i \quad (54)$$

$$\begin{aligned} C_{k+1}^{\bar{i}} &= \sum_{j=1, j \neq i}^{M_{k+1|k}} (r_j \cdot v_j - r_j^2 \cdot s_j s_j^T) \\ &= C_{k+1} - (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) \end{aligned} \quad (55)$$

## APPENDIX B

The PHD of the multi-target distribution  $f_{k+1|k}^n(W)$  is by definition

$$\begin{aligned} D_{k+1|k}^n(\mathbf{x}) &= \int f_{k+1|k}^n(\{\mathbf{x}\} \cup W) \delta W \\ &= \frac{1}{\pi_{k+1|k}(n)} \int_{|W|=n-1} f_{k+1|k}(\{\mathbf{x}\} \cup W) \delta W \end{aligned} \quad (56)$$

$$\begin{aligned} &= \frac{1}{\pi_{k+1|k}(n)} \cdot \frac{1}{(n-1)!} \\ &\quad \times \int f_{k+1|k}(\{\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}) d\mathbf{y}_1 \dots d\mathbf{y}_{n-1} \end{aligned} \quad (57)$$

$$\begin{aligned} &= \frac{1}{\pi_{k+1|k}(n)} \cdot \frac{1}{(n-1)!} \\ &\quad \times \sum_{i=1}^{M_{k+1|k}} r_i \cdot p_i(\mathbf{x}) \cdot (n-1)! \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \end{aligned} \quad (58)$$

$$\begin{aligned} &= \frac{1}{\pi_{k+1|k}(n)} \sum_{i=1}^{M_{k+1|k}} r_i \cdot p_i(\mathbf{x}) \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \end{aligned} \quad (59)$$

where  $\pi_{k+1|k}^{\bar{i}}(n)$  is the cardinality distribution excluding the  $i^{\text{th}}$  component of the multi-Bernoulli RFS. Similarly we can find the second factorial density as

$$\begin{aligned} D_{k+1|k}^n(\{\mathbf{x}_1, \mathbf{x}_2\}) &= \frac{1}{\pi_{k+1|k}(n)} \\ &\quad \times \sum_{i=1}^{M_{k+1|k}} \sum_{j=1, i \neq j}^{M_{k+1|k}} r_i \cdot p_i(\mathbf{x}_1) \cdot r_j \cdot p_j(\mathbf{x}_2) \cdot \pi_{k+1|k}^{\bar{i}, \bar{j}}(n-2) \end{aligned}$$

where  $\pi_{k+1|k}^{\bar{i}, \bar{j}}(n)$  is the cardinality distribution excluding the  $i^{\text{th}}$  and  $j^{\text{th}}$  component of the multi-Bernoulli RFS. Applying Campbell's theorem we get

$$\begin{aligned} \mu_{k+1}^n &= \int g(\mathbf{x}) \cdot D_{k+1|k}^n(\mathbf{x}) d\mathbf{x} \\ &= \int \frac{g(\mathbf{x})}{\pi_{k+1|k}(n)} \left( \sum_{i=1}^{M_{k+1|k}} r_i \cdot p_i(\mathbf{x}) \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \right) d\mathbf{x} \\ &= \frac{1}{\pi_{k+1|k}(n)} \sum_{i=1}^{M_{k+1|k}} r_i \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \cdot s_i \end{aligned}$$

$$\begin{aligned} C_{k+1}^n &= \frac{1}{\pi_{k+1|k}(n)} \left( \sum_{i=1}^{M_{k+1|k}} r_i \cdot \pi_{k+1|k}^{\bar{i}}(n-1) \cdot v_i \right. \\ &\quad \left. + \sum_{i \neq j} r_i \cdot r_j \cdot \pi_{k+1|k}^{\bar{i}, \bar{j}}(n-2) \cdot s_i s_j^T \right) - \mu_{k+1}^n (\mu_{k+1}^n)^T \end{aligned}$$

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