Distributed Optimization over a Network

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The recent advances in wired and wireless technology lead to the emergence of large-scale networks:

- Internet
- Mobile ad-hoc networks
- Wireless sensor networks

The advances gave rise to new network applications including:

- Decentralized network operations including resource allocation, coordination, learning, estimation
- Database networks
- Social and economic networks

As a result, there is a necessity to develop new models and tools for the design and performance analysis of such large complex dynamics systems.
New Application Challenges

- **Lack of central “authority”**
  - Centralized network architecture is **not possible**
    - Size of the network / Proprietary issues
  - Sometimes centralized architecture is **not desirable**
    - Security issues / Robustness to failures

- **Network dynamics**
  - Mobility of the network
    - The agent spatio-temporal dynamics
    - Network connectivity structure is varying in time
  - Time-varying network
    - The network itself is evolving in time

- The challenge is to design algorithms to support efficient operations over such networks.
Trends: Areas of Research

- **Network aspects**
  - Consensus, information diffusion, opinion dynamics
    - Stability, characterization of equilibria, bounds on convergence rate/time

- **Optimization issues**
  - Distributed algorithms for optimization over networks
    - Different network performance measures
    - Synchronous/asynchronous implementations
    - Different information diffusion/exchange protocols

- **Uncertainty issues in distributed systems**
  - Communication noise and delay (network dependent)
  - Uncertain measurements (application dependent)
  - Computational errors (algorithm dependent)
Distributed Computational Model: Self-organized Agents

The model consists of a network of computing agents (nodes) that cooperate in order to optimize a network-wide objective function

- Agents can communicate only with immediate neighbors in the network
- Agents have individual objective functions that they do not "reveal" to each other
- Agents cooperate (share info.) with their neighbors

Distributed load-balancing in a network

Uplink power control in mobile cellular net

Network objective:

$$\text{minimize}_{x \in \mathbf{X}} \max_{1 \leq i \leq m} f_i(x)$$
General Model

- Network of \( m \) agents represented by an undirected graph \(([m], \mathcal{E})\) where \([m] = \{1, \ldots, m\}\) and \(\mathcal{E}\) is the edge set

- Each agent \( i \) has an objective function \( f_i(x) \) known to that agent only

- Common constraint (closed convex) set \( X \) known to all agents

The problem can be:

\[
\text{minimize } \sum_{i=1}^{m} f_i(x) \text{ subject to } x \in X \subseteq \mathbb{R}^n
\]

or

\[
\text{minimize } \max_{1 \leq i \leq m} f_i(x) \text{ subject to } x \in X \subseteq \mathbb{R}^n
\]
How Agents Manage to Optimize

\[
\begin{align*}
& \text{minimize } \sum_{i=1}^{m} f_i(x) \quad \text{subject to } x \in X \subseteq \mathbb{R}^n \\
& \text{minimize } \max_{1 \leq i \leq m} f_i(x) \quad \text{subject to } x \in X \subseteq \mathbb{R}^n
\end{align*}
\]

- Each agent \( i \) will generate its own estimate \( x^i(t) \) of an optimal solution to the problem

- Each agent will update its estimate \( x^i(t) \) by performing two steps:
  - Consensus-like step (mechanism to align agents estimates toward a common point)
  - Local gradient-based step (to minimize its own objective function)


Consensus Problem

Consider a connected network of \( m \)-agent, each knowing its own scalar value \( x_i(0) \) at time \( t = 0 \).

The problem is to design a distributed and local algorithm ensuring that the agents agree on the same value \( x \), i.e.,

\[
\lim_{t \to \infty} x_i(t) = x \quad \text{for all } i.
\]

Leaderless Heading Alignment

A system of autonomous agents are moving in the plane with the same speed but with different headings [Vicsek 95, Jadbabaie et al. 03]

The objective is to design a local protocol that will ensure the alignment of agent headings.
Consensus Algorithm

Each agent combines its estimate $x_i(t)$ with the estimates $x_j(t)$ received from its neighbors

$$x_i(t + 1) = \sum_{j \in N_i} a_{ij} x_j(t) \quad \text{for all } i.$$  

where $N_i$ is the set of neighbors of agent $i$ (including itself)

$$N_i = \{ j \in [m] \mid (i, j) \in E \}$$  

and $a_{ij} \geq 0$ is a weight that agent $i$ assigns to the information coming from its neighbor $j \in N_i$.

The weights $\{a_{ij}, j \in N_i\}$ sum to 1, i.e., $\sum_{j \in N_i} a_{ij} = 1$ for all agents $i$.

Introducing the values $a_{ij} = 0$ when $j \not\in N_i$, the consensus algorithm can be written as:

$$x_i(t + 1) = \sum_{j=1}^{m} a_{ij} x_j(t)$$

where

$$a_{ij} \geq 0 \quad \text{with } a_{ij} = 0 \text{ when } j \not\in N_i$$

$$\sum_{j=1}^{m} a_{ij} = 1$$
Distributed Optimization Algorithm

minimize $\sum_{i=1}^{m} f_i(x)$ subject to $x \in X \subseteq \mathbb{R}^n$

- At time $t$, each agent $i$ has its own estimate $x^i(t)$ of an optimal solution to the problem

- At time $t + 1$, agents communicate their estimates to their neighbors and update by performing two steps:
  - Consensus-like step to mix their own estimate with those received from neighbors
    $$w^i(t + 1) = \sum_{j=1}^{m} a_{ij} x^j(t) \quad (a_{ij} = 0 \text{ when } j \notin N_i)$$
  - Followed by a local gradient-based step
    $$x^i(t + 1) = \Pi_X [w^i(t + 1) - \alpha(t) \nabla f_i(w^i(t + 1))]$$

where $\Pi_X[y]$ is the Euclidean projection of $y$ on $X$, $f_i$ is the local objective of agent $i$ and $\alpha(t) > 0$ is a stepsize
Intuition Behind the Algorithm: It can be viewed as a consensus steered by a "force":

\[ x^i(t + 1) = w^i(t + 1) + \left( \prod_X [w^i(t + 1) - \alpha(t) \nabla f_i(w^i(t + 1))] - w^i(t + 1) \right) \]

\[ = w^i(t + 1) + \left( \prod_X [w^i(t + 1) - \alpha(t) \nabla f_i(w^i(t + 1))] - \prod_X [w^i(t + 1)] \right) \]

small stepsize \( \alpha(t) \)

\[ \approx w^i(t + 1) - \alpha(t) \nabla f_i(w^i(t + 1)) \]

\[ = \sum_{j=1}^{m} a_{ij} x^j(t) - \alpha(t) \nabla f_i \left( \sum_{j=1}^{m} a_{ij} x^j(t) \right) \]

Matrices \( A \) that lead to consensus, also yield convergence of an optimization algorithm.
Convergence Result for Static Network

Convex Problem: Let $X$ be closed and convex, and each $f_i : \mathbb{R}^n \to \mathbb{R}$ be convex with bounded (sub)gradients over $X$. Assume the problem $\min_{x \in X} \sum_{i=1}^{m} f_i(x)$ has a solution.

Stepsize Rule: Let the stepsize $\alpha(t)$ be such that $\sum_{t=0}^{\infty} \alpha(t) = \infty$ and $\sum_{t=0}^{\infty} \alpha^2(t) < \infty$.

Network: Let the graph $(\mathcal{V}, E)$ be directed and strongly connected. Let the matrix $A = [a_{ij}]$ of agents' weights be doubly stochastic. Then,

$$\lim_{t \to \infty} x^i(t) = x^* \quad \text{for all } i,$$

where $x^*$ is a solution of the problem.

Proof Outline:
Use $\sum_{i=1}^{m} \|x^i(t) - x^*\|^2$ as a Lyapunov function, where $x^*$ is a solution to the problem.

Due to convexity and (sub)gradient boundedness, we have

$$\sum_{i=1}^{m} \|x^i(t+1) - x^*\|^2 \leq \sum_{i=1}^{m} \|w^i(t+1) - x^*\|^2 - 2\alpha(t) \sum_{i=1}^{m} (f_i(w^i(t+1)) - f_i(x^*)) + \alpha^2(t)C^2$$
By $w^i(t + 1) = \sum_{j=1}^{m} a_{ij} x^j(t)$ and the doubly stochasticity of $A$, we have

$$\sum_{i=1}^{m} \|x^i(t+1) - x^*\|^2 \leq \sum_{j=1}^{m} \|x^j(t) - x^*\|^2 - 2\alpha(t) \sum_{i=1}^{m} (f_i(w^i(t + 1)) - f_i(x^*)) + \alpha^2(t)C^2$$

Thus, letting $s(t + 1) = \frac{1}{m} \sum_{i=1}^{m} x^i(t + 1)$ we see

$$\sum_{i=1}^{m} \|x^i(t + 1) - x^*\|^2 \leq \sum_{j=1}^{m} \|x^j(t) - x^*\|^2 - 2\alpha(t) \sum_{i=1}^{m} (f_i(s(t + 1)) - f_i(x^*))$$

$$+ 2\alpha(t) \sum_{i=1}^{m} (f_i(s(t + 1)) - f_i(w^i(t + 1))) + \alpha^2(t)C^2$$

Letting $F(x) = \sum_{i=1}^{m} f_i(x)$ and using (sub)gradient boundedness, we find

$$\sum_{i=1}^{m} \|x^i(t + 1) - x^*\|^2 \leq \sum_{j=1}^{m} \|x^j(t) - x^*\|^2 - 2\alpha(t) \left[ F(s(t + 1)) - F(x^*) \right] \geq 0$$

$$+ 2\alpha(t)C \sum_{i=1}^{m} \|s(t + 1) - w^i(t + 1)\| + \alpha^2(t)C^2$$

We can see $\sum_{t=0}^{\infty} \alpha(t)C \sum_{i=1}^{m} \|s(t + 1) - w^i(t + 1)\| < \infty$

The result would hold if we can show $\|s(t + 1) - w^i(t + 1)\| \to 0$ as $t \to \infty$ for all $i$
The trouble is in showing $\|s(t+1) - w^i(t+1)\| \to 0$ as $t \to \infty$ for all $i$, which is exactly where the **network impact** is – rate of convergence of $A^t$ to its limit is needed.

When the network is connected, the matrices $A^t$ converge to the matrix $\frac{1}{m}11'$, as $t \to \infty$.

The convergence rate is

$$\left| [A^t]_{ij} - \frac{1}{m} \right| \leq q^t, \text{ where } q \in (0, 1)$$

We have for arbitrary $0 \leq \tau < t$

$$x^i(t+1) = w^i(t+1) + \left( \prod_x [w^i(t+1) - \alpha(t) \nabla f_i(w^i(t+1))] - w^i(t+1) \right)\frac{e^i(t)}{e^i(t)}$$

$$= \sum_{j=1}^m a_{ij} x^j(t) + e^i(t) = \cdots$$

$$= \sum_{j=1}^m [A^{t+1-\tau}]_{ij} x^j(\tau) + \sum_{k=\tau+1}^t \sum_{j=1}^m [A^k]_{ij} e_j(t-k) + e^i(t)$$

Similarly, for $s(t+1) = \frac{1}{m} \sum_{i=1}^m x^i(t+1)$ we have

$$s(t+1) = s(t) + \frac{1}{m} \sum_{j=1}^m e^j(t) = \cdots = \sum_{j=1}^m \frac{1}{m} x^j(\tau) + \sum_{k=\tau+1}^t \sum_{j=1}^m \frac{1}{m} e_j(t-k) + \sum_{j=1}^m \frac{1}{m} e^j(t)$$
Thus,
\[ ||x^i(t+1) - s(t+1)|| \leq q^{t+1-\tau} \sum_{j=1}^{m} ||x^j(\tau)|| + \sum_{k=\tau+1}^{t} \sum_{j=1}^{m} q^k ||e_j(t-k)|| + \sum_{j=1}^{m} \frac{1}{m} ||e^i(t)|| + ||e^i(t)|| \]

Since
\[ e^i(t) = \prod_X [w^i(t+1) - \alpha(t) \nabla f_i(w^i(t+1))] - w^i(t+1) \]
we have
\[ ||e^i(t)|| \leq \alpha(t) C \]

Hence
\[ ||x^i(t+1) - s(t+1)|| \leq q^{t+1-\tau} \sum_{j=1}^{m} ||x^j(\tau)|| + mC \sum_{k=\tau+1}^{t} q^k \alpha(t-k) + 2C \alpha(t) \]

By choosing \( \tau \) such that \( ||e(t)|| \leq \epsilon \) for all \( t \geq \tau \) and then, using some properties of the sequences involved in the above relation, we show
\[ ||x^i(t+1) - s(t+1)|| \rightarrow 0 \]
which in view of \( x^i(t+1) = w^i(t+1) + e^i(t) \) and \( e^i(t) \rightarrow 0 \) implies
\[ ||w^i(t+1) - s(t+1)|| \rightarrow 0 \]
Extension to Time-varying Networks

- Consensus-like step to mix their own estimate with those received from neighbors

\[ w^i(t + 1) = \sum_{j=1}^{m} a_{ij}(t)x^j(t) \quad (a_{ij}(t) = 0 \text{ when } j \notin N_i(t)) \]

- Followed by a local gradient-projection step

\[ x^i(t + 1) = \Pi_X[w^i(t + 1) - \alpha(t)\nabla f_i(w^i(t + 1))] \]

For convergence, some conditions on the weight matrices \( A(t) = [a_{ij}(t)] \) are needed.

**Convergence Result for Time-varying Network\(^*\):**

Let the problem be convex, \( f_i \) have bounded (sub)gradients on \( X \), and \( \sum_{t=0}^{\infty} \alpha(t) = \infty \) and \( \sum_{t=0}^{\infty} \alpha^2(t) < \infty \). Let the graphs \( G(t) = ([m], \mathcal{E}(t)) \) be directed and strongly connected, and the matrices \( A(t) \) be such that \( a_{ij}(t) = 0 \) if \( j \notin N_i(t) \), while \( a_{ij}(t) \geq \gamma \) whenever \( a_{ij}(t) > 0 \), where \( \gamma > 0 \). Also assume that \( A(t) \) are doubly stochastic\(^\dagger\). Then,

\[ \lim_{t \to \infty} x^i(t) = x^* \quad \text{for all } i, \]

where \( x^* \) is a solution of the problem.


Other Extensions

\[ w^i(t+1) = \sum_{j=1}^{m} a_{ij}(t)x^j(t) \quad (a_{ij}(t) = 0 \text{ when } j \notin N_i(t)) \]

\[ x^i(t+1) = \prod_X [w^i(t+1) - \alpha(t)\nabla f_i(w^i(t+1))] \]

Extensions include

- Gradient directions \( \nabla f_i(w^i(t+1)) \) can be erroneous

\[ x^i(t+1) = \prod_X [w^i(t+1) - \alpha(t)(\nabla f_i(w^i(t+1) + \varphi_i(t+1))] \]

- The set \( X \) can be \( X = \bigcap_{i=1}^{m} X_i \) where each \( X_i \) is a private information of agent \( i \)
  Nedić, Ozdaglar, and Parrilo 2010, Srivastava‡ and Nedić 2011

\[ x^i(t+1) = \prod_{X_i} [w^i(t+1) - \alpha(t)\nabla f_i(w^i(t+1))] \]

- The links can be noisy i.e., \( x^j(t) \) is sent to agent \( i \), but the agent receives \( x^j(t) + \epsilon_{ij}(t) \)
  Srivastava and Nedić 2011

- The updates can be asynchronous - edge-set \( \mathcal{E}(t) \) is random
  Ram, Nedić, and Veeravalli, Nedić 2011

- Different sum-based functional structures [Ram, Nedić, and Veeravalli 2012]

‡Uses different weights
**Application to Data Classification**

Given a set of data points \( \{(z_j, y_j), j = 1, \ldots, p\} \), find a vector \((x, u)\) that

\[
\text{minimizes } \frac{\lambda}{2} \|x\|^2 + \sum_{j=1}^{p} \max\{0, 1 - y_j(x'z_j + u)\}
\]

Suppose that the data is distributed at \( m \) locations, with each location having data points \( \{(z_\ell, y_\ell), \ell \in S_i\} \), with \( S_i \) being the index set

The problem can be written as:

\[
\text{minimize } \sum_{i=1}^{m} \left( \frac{\lambda}{2m} \|x\|^2 + \sum_{\ell \in S_i} \max\{0, 1 - y_\ell(x'z_\ell + u)\} \right) \\
\text{over } x = (x, u) \in \mathbb{R}^n \times \mathbb{R} \quad f_i(x)
\]

The algorithm has the form:

\[
w^i(t + 1) = x^i(t) - \eta(t) \sum_{j=1}^{m} r_{ij}x^j(t) \quad (r_{ij} = 0 \text{ when } j \notin N_i)
\]

\[
x^i(t + 1) = w^i(t + 1) - \alpha(t) g_i(w^i(t + 1)) \quad \text{subgradient}
\]

Case with perfect communications

Illustration uses a simple graph of 4 nodes organized in a ring-network

\[ \lambda = 6 \]
\[ \alpha(t) = \frac{1}{t} \]
\[ \eta(t) = 0.8 \]

After 20 iterations

After 500 iterations
Case with imperfect communications

\[
\begin{align*}
\text{minimize } & \sum_{i=1}^{m} \left( \frac{\lambda}{2m} \|x\|^2 + \sum_{\ell \in S_i} \max\{0, 1 - y_{\ell}(x'z_{\ell} + u)\} \right) \\
& \text{over } x = (x, u) \in \mathbb{R}^n \times \mathbb{R} \\
& f_i(x)
\end{align*}
\]

\[
\begin{align*}
  w^i(t + 1) &= x(t) - \eta(t) \sum_{j=1}^{m} r_{ij}(x^j(t) + \xi_{ij}(t)) \\
  \text{with } w_{ij} &= 0 \text{ when } j \notin N_i, \eta(t) > 0 \text{ is a noise-damping stepsize}
\end{align*}
\]

\[
\begin{align*}
  x^i(t + 1) &= w^i(t + 1) - \alpha(t) g_i(w^i(t + 1))
\end{align*}
\]

Noise-damping stepsize \(\eta(t)\) has to be coordinated with sub-gradient related stepsize \(\alpha(t)\)

\[
\begin{align*}
  \sum_t \alpha(t) &= \infty, \quad \sum_t \alpha^2(t) < \infty \\
  \sum_t \eta(t) &= \infty, \quad \sum_t \eta^2(t) < \infty \\
  \sum_t \alpha(t)\eta(t) &< \infty, \quad \sum_t \frac{\alpha^2(t)}{\eta(t)} < \infty
\end{align*}
\]
Case with imperfect communications

Illustration uses a simple graph of 4 nodes organized in a ring-network

\[
\begin{align*}
\lambda &= 6 \\
\alpha(t) &= \frac{1}{t} \\
\eta(t) &= \frac{1}{t^{0.55}}
\end{align*}
\]

After 1 iteration

After 500 iterations
Minimizing Max of Agents’ Objectives

Network objective: minimize $\max_i f_i(x)$ over $x \in X \subseteq \mathbb{R}^n$

Makes sense when "fair network resource-utilization" is desired

Use epi-graph reformulation of the problem

$$\begin{align*}
\text{minimize} & \quad \eta \\
\text{subject to} & \quad f_i(x) \leq \eta \quad \text{for all } i = 1, \ldots, m, \; x \in X, \; \eta \in \mathbb{R}
\end{align*} \quad (1)$$

Under Slater condition (satisfied here) the strong duality holds for problem (1) and its dual

$$\begin{align*}
\text{maximize} & \quad q(\mu) \quad \text{over } \mu \geq 0, \mu \in \mathbb{R}^m, \\
\text{where} & \quad q(\mu) = \min_{x \in X, \eta \in \mathbb{R}} \left( \eta + \sum_{i=1}^{m} \mu_i (f_i(x) - \eta) \right)
\end{align*}$$

So the problem can be solved by using a \textbf{primal-dual algorithm} or a \textbf{penalty approach}.

Consider penalty approach: problem (1) is replaced with an equivalent "penalized" problem

$$\begin{align*}
\text{minimize} & \quad F(x, \eta) = \eta + \sum_{i=1}^{m} c_i (f_i(x) - \eta) \\
\text{subject to} & \quad (x, \eta) \in X \times \mathbb{R}
\end{align*}$$

where $c_i > 1$ for all $i$. 
Distributed Algorithm for Min-Max Optimization

\[
\begin{align*}
\text{minimize} & \quad F(x, \eta) = \sum_{i=1}^{m} \left( \frac{\eta}{m} + c_i (f_i(x) - \eta) \right) \\
\text{subject to} & \quad (x, \eta) \in X \times \mathbb{R} \\
\text{where} & \quad c_i > 1 \text{ for all } i.
\end{align*}
\]

\[F_i(x, \eta) = \left( \frac{1}{m} - 1 \right) \eta + c_i f_i(x)\]

Observations:

- Each agent can choose its own \(c_i\), as long as \(c_i > 1\)

- Every agent has to know \(m\)

- \(\nabla_x F_i(x, \eta) = c_i \nabla f_i(x)\) and \(\nabla_\eta F_i(x, \eta) = \frac{1}{m} - 1\)
Example: Uplink Power Control

- $m$ mobile users (MU) communicate with respective base stations (BS)
- $p_i$ - power used by $i^{th}$ MU to communicate with $i^{th}$ BS
- $p = (p_1, \ldots, p_m)$ power-allocation vec.
- $h_{i,j}$ - channel coefficient for MU $j$ and BS $i$
- $h_i = (h_{i1}, \ldots, h_{im})$ channel coef. vec. for $i$
- $\sigma_i^2$ - receiver noise variance
- SINR at BS $i$ is given by
  $$\gamma_i(p, h_i) = \frac{p_i h_{i,i}^2}{\sigma_i^2 + \sum_{j \neq i} p_j h_{j,j}^2},$$
- $U_i(\gamma_i(p, h_i))$ is the utility for BS $i$ based on SINR
- $V(p_i)$ is a cost function penalizing excessive power
- We are interested in computing the min-max fair allocation
  $$\min_{p \in \Pi} \max_{i \in V} \left[ V(p_i) - U_i(\gamma_i(p, h_i)) \right], \quad \Pi = \{p \in \mathbb{R}^m \mid 0 \leq p_i \leq p_{\text{max}} \text{ for all } i\}$$
**Example: Uplink Power Control**

- Generally a non-convex problem
- Logarithmic utility $U_i(y) = \ln(y)$
- Linear power cost $V(p_i) = a p_i$, $a > 0$
- The coordinate transformation $p_i = e^{x_i}$ makes the problem a convex optimization problem (in $x$)

- The resulting convex problem is

$$
\min_{x \in X} \max_{i \in V} f_i(x),
$$

$$
f_i(x) = \ln \left( \sigma_i^2 h_{i,i}^{-2} e^{-x_i} + \sum_{j \neq i} h_{i,i}^{-2} h_{j,i}^2 e^{x_j-x_i} \right) + V(e^{x_i})
$$

and $X = \{ x \mid x_i \leq \ln(p_{\text{max}}) \text{ for all } i \}$. 

The final iterate values after 2000 iterations of the algorithm for step size $\alpha_k = 50 / k^{0.65}$. 

![Graph showing comparison of different optimization algorithms](image-url)
Conclusions

• Considered algorithms for distributed optimization over network

• Illustrated them on data classification problem

• Considered dynamic TU games over networks
  • Dynamic in the game and in the player’s network connectivity

• Discussed distributed allocation algorithms that converge to an allocation in the core of the limiting game