Multi-Bernoulli filter and hybrid multi-Bernoulli CPHD filter for superpositional sensors

Technical Report

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***Abstract*—Superpositional sensor model can characterize the observations in many different applications such as radio frequency tomography, acoustic sensor network based tracking and wireless communications. In this paper we present two filters based on the random finite set (RFS) theory - the multi-Bernoulli filter and its variant the hybrid multi-Bernoulli CPHD filter - for superpositional sensors. We provide derivations for the filter update equations which are based on propagating the conditional probability hypothesis density (PHD). The conditional PHD is defined for the individual Bernoulli components of the multi-Bernoulli RFS and for the independent and identically distributed cluster (IIDC) RFS. Computationally tractable update equations are derived by assuming the sensor noise to be Gaussian. The filters are used for multitarget tracking in simulated radio frequency (RF) tomography application.**

***Index Terms*—Random finite set, CPHD filter, multi-Bernoulli filter, superpositional sensors, radio frequency tomography, multitarget tracking.**

I. Introduction

We study the problem of sequential multitarget state estimation from noisy sensor observations available at regular time intervals. Many problems in engineering can be modeled using this framework examples of which include multitarget localization and tracking, wireless channel estimation and weather prediction. The traditional approach has been to model the state and observations as random vectors. This approach has drawbacks since it cannot efficiently model the changing multitarget state dimension and observation dimension. To counter this drawback, Mahler [1], [2] has formulated the problem of sequential state estimation using the random finite set (RFS) framework. In this framework the multitarget state and the observations are modeled as realizations of random sets. The corresponding finite set statistics (FISST) that has been developed allows us to build approximate state propagation and estimation algorithms which are computationally feasible.

Most of the initial research on RFS theory focused on specific type of sensors, called as standard sensors [3]. The standard sensor observation model can be characterized as follows: (i) each target causes either one or no measurement; and (ii) each measurement is either caused by a single target or clutter. Examples of this category of sensors are range sensors, bearing sensors, radar and sonar. We are interested in a different but important class of sensors, called as superpositional sensors [3]. The superpositional sensor observation model has following characteristics: (i) each target can contribute to any number of measurements; (ii) each measurement is potentially affected by multiple targets in an additive fashion; and (iii) measurements are not independent. Many sensors belong to the category of superpositional sensors. Examples include direction-of-arrival sensors for linear antenna arrays [4], antenna arrays in multi-user detection for wireless communication networks [5], multipath channel modeling in MIMO-OFDM channels [6], acoustic amplitude sensors [7], and radio frequency tomographic tracking systems [8]. In literature studying the standard sensors, both the multitarget state and the observations are modeled as random finite sets. In this work we model the multitarget state as a RFS and the observations are modeled as random vectors.

Various multitarget state estimation filters have been proposed using the FISST framework. These filters differ in the assumptions they make about the underlying multitarget state. The PHD filter [9] assumes the state to be realization of a RFS with Poisson multitarget distribution. It has the advantage that a single PHD function is sufficient to characterize the complete distribution and is computationally easy to track. The CPHD filter [10] is an improvement of the PHD filter and it uses the IIDC RFS to model the state. A PHD function and a cardinality distribution are required to characterize the IIDC multitarget distribution. The additional cardinality information improves performance of the CPHD filter. Though technically superior to the PHD filter, the CPHD filter still utilizes a single density function to represent the multiple states and this can reduce its accuracy of state estimation. It also requires an additional clustering step when implemented using a particle filter.

The multi-Bernoulli filter [2] can alleviate the drawbacks of PHD and CPHD filter and models each target state with a scalar existence probability and a state density function. This allows accurate state representation and also provides easy track maintenance without the need for clustering step. The multi-Bernoulli filter provides accurate state estimation but it is less robust to arrival of new targets and disappearance of existing targets. This issue can be addressed by using a hybrid multi-Bernoulli CPHD filter in which the new targets are modeled as realizations of the IIDC RFS while the existing targets are modeled using the multi-Bernoulli RFS.

In this paper we provide the derivation of the multi-Bernoulli filter [11] and the hybrid multi-Bernoulli CPHD filter update equations. The derivation relies on defining the quantity conditional PHD and performing approximate Bayes' update for the different conditional PHD terms. The cardinality distribution of the IIDC component is also propagated. We provide pseudo code for auxiliary particle filter implementa-

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tions of the filters. A numerical study is performed to assess the filter performance in comparison with the CPHD filter in multitarget tracking application.

The remainder of the paper is organized as follows. Section I-A provides a summary of the research done on RFS theory based filters and superpositional sensors. The multitarget state estimation problem formulation is given in Section I-B. A brief review of the RFS theory and the related statistics is presented in Section II. The concept of conditional PHD is introduced in Section III and its update equation is derived. The multi-Bernoulli and the hybrid multi-Bernoulli CPHD filter update equations are derived in Section IV. Auxiliary particle filter implementation of these filters are presented in Section V. Section VI compares the multi-Bernoulli and the hybrid multi-Bernoulli CPHD filter with the CPHD filter in the context of multitarget tracking. Simulation results for radio frequency tomography based tracking are presented. Finally we provide conclusions in Section **??**.

A. Literature review

A detailed description of the RFS theory, FISST statistics and the different filters based on it can be found in [2]. The difference between the standard sensor model and the superpositional sensor model was first explicitly stated by Mahler in [3]. The PHD and CPHD filter equations for the standard sensor model were first derived in [9] and [10] respectively. The PHD filter for superpositional sensors was derived by Thouin et al. in [12] [1]. The CPHD filter for superpositional sensors was derived by Mahler and El-Fallah in [14]. An auxiliary particle filter implementation of the PHD and CPHD filters for superpositional sensors was presented in [15] along with their numerical studies in simulated multitarget tracking applications.

The multi-Bernoulli filter for standard sensors was first proposed by Mahler in [2]. The cardinality bias in the original filter formulation was subsequently corrected by Vo et al. in [16]. Particle filter implementations of the multi-Bernoulli filter are discussed in [16], [17]. Under the assumptions that the likelihood has a separable form, the multi-Bernoulli filter has been used for estimation and detection of objects from images in track-before-detect applications [17]. The separable likelihood assumption is valid for non-overlapping targets. Convergence analysis of the sequential Monte Carlo (SMC) implementations of the multi-Bernoulli filters are performed by Lian et al. in [18]. The multi-Bernoulli filter for superpostional sensors was proposed by Nannuru and Coates [11]. A particle filter implementation of this filter is presented by the same authors in [19].

We make a note about the relation between the particle implementation of the multi-Bernoulli filter and the multiple particle filter (MPF) proposed by Djuric et al. in [20]. Both of them use one particle filter per target but the number of targets is assumed to be fixed and known in the MPF implementation. The multi-Bernoulli filter automatically keeps track of the changing number of targets and maintains appropriate number of particle filters required. The weight update in MPF propagates the marginal posterior for each target conditional on all the remaining target states whereas the weight update in multi-Bernoulli filter propagates the conditional PHD.

A hybrid between the multi-Bernoulli filter and the PHD filter was proposed by Williams [21], [22] for multitarget tracking applications. The author uses a Poisson RFS to model new targets and targets with low probability of existence. This results in fast track initiation and use of fewer Bernoulli components. Pollard et al. in [23] have used a hybrid combination of multiple hypothesis tracking (MHT) filter and the Gaussian mixture CPHD (GM-CPHD) filter for multitarget tracking. The GM-CPHD filter provides robust cardinality estimate of the multitarget state which is complemented by accurate state estimates from the MHT filter. The combination of MHT and PHD filter has been proposed by Panta et al. to obtain track-valued estimates [24]. The authors use the PHD filter as a clutter filter by using its output to gate the input for the MHT filter.

B. Problem formulation

In this paper we study the superpositional sensors in the context of multitarget tracking problem. The multi target state is the set $X_k = \{\mathbf{x}_{k,1}, \mathbf{x}_{k,2}, \dots, \mathbf{x}_{k,n_k}\}$ where $\mathbf{x}_{k,i}$, $i = 1, 2, \dots, n_k$ are the single target state vectors of the $n_k \geq 0$ targets present at time step k. The single target state dimension is $n_{\mathbf{x}}$, so $\mathbf{x}_{k,i} \in R^{n_{\mathbf{x}}}$ $\forall i$. The targets move independently and their motion is governed by the Markovian transition kernel $t_{k|k-1}(\mathbf{x}_{k,i}|\mathbf{x}_{k-1,i}, \mathbf{u}_k)$ where $\mathbf{u}_k$ is the Gaussian noise vector. This state information is hidden but we have access to the observation vector $\mathbf{z}_k \in R^{n_{\mathbf{z}}}$ at time step k. The observations are related to the multitarget state through the likelihood function $h_{\mathbf{z}_k}(X_k)$. Let $Z^{[k]} = [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k]$ be the collection of all the observation vectors up to time k. The multitarget tracking problem is to estimate the posterior multitarget state density $p(X_k|Z^{[k]})$ at each time step k.

C. Superpositional sensor model

The likelihood function under the superpostional model assumption has the following form

$$\begin{aligned} h_{\mathbf{z}_k}(X_k) &= h_{\mathbf{z}_k}\left(r(X_k)\right) \\ &= h_{\mathbf{z}_k}\left(\sum_{\mathbf{x}\in X_k} g(\mathbf{x})\right) \end{aligned} \tag{1}$$

where $h_{\mathbf{z}_k}$ is the real-valued likelihood function and g and r are (potentially non-linear) functions mapping to vectors of reals. When the sensor observation noise is Gaussian with zero mean and covariance matrix Σ_r, the likelihood takes the form

$$h_{\mathbf{z}_k}(X_k) = \mathcal{N}_{\Sigma_r}(\mathbf{z}_k - r(X_k)). \tag{2}$$

where $\mathcal{N}_{\Sigma}(\mathbf{z})$ denotes the Gaussian density function with zero mean and covariance matrix Σ evaluated at $\mathbf{z}$.

II. Random finite sets

Random sets are extension of the concept of random variables and random vectors. While random vectors are of

[1] An error in the main update equation of this filter was corrected in an errata [13]; the correct equations were also presented in [14], [15].

a predefined dimension and they have an ordering of their elements, random set can have uncertainty in the set dimension and there is no preferred ordering between the elements of the set.

We can associate a probability density function with a RFS. The integral of the probability density function is one. Since this probability density function is defined for a RFS, the integral is the set integral defined as follows

$$\int f(W)\delta W = f(\phi) + \sum_{n=1}^{\infty}\frac{1}{n!}\int f(\{\mathbf{w}_1,\dots,\mathbf{w}_n\})d\mathbf{w}_1\dots d\mathbf{w}_n$$
$$= f(\phi) + \sum_{n=1}^{\infty}\frac{1}{n!}\int f(W_n)dW_n = 1 \quad (3)$$

where ϕ is the empty set. The notation δW denotes set integration and the notation dW_n denotes the standard integration. We have used the abbreviated notation $dW_n = d\mathbf{w}_1\dots d\mathbf{w}_n$ for brevity. The associated cardinality distribution of the RFS is given by

$$Prob(|W| = n) = \pi(n) = \frac{1}{n!}\int_{|W|=n} f(W)\delta W \quad (4)$$

Since addition operation is not naturally defined on sets, defining the expectation of a RFS in traditional manner is not possible. An important and useful statistic of the RFS which can be defined using a modified definition of first moment [2] is the probability hypothesis density (PHD). The PHD of a RFS with a probability density $f(W)$ is defined as follows

$$D(\mathbf{x}) = \int f(\{\mathbf{x}\}\cup W)\delta W \quad (5)$$

Similarly the second factorial moment is defined as

$$D(\{\mathbf{x}_1,\mathbf{x}_2\}) = \int f(\{\mathbf{x}_1,\mathbf{x}_2\}\cup W)\delta W \quad (6)$$

We now give some examples of random finite sets.

A. IID cluster RFS

An independent and indentically distributed cluster RFS is specified using an arbitrary cardinality distribution and a density function. It has the following multitarget distribution

$$f^c(X) = |X|\cdot \pi^c(|X|)\cdot q_c^X \quad (7)$$

$$q_c^X = \begin{cases} 1, & \text{if } X = \phi \\ \prod_{\mathbf{x}\in X} q_c(\mathbf{x}) & \text{otherwise.}\end{cases} \quad (8)$$

where $|X|$ denotes the cardinality of the set X; $\pi^c(n)$ and $q_c(\mathbf{x})$ are the cardinality distribution function and the density function respectively. Let $G(t)$ denote the probability generating function of the cardinality distribution $\pi^c(n)$ and let $G^{(i)}(t)$ denote its i^{th} derivative

$$G(t) = \sum_{n=0}^{\infty} t^n\pi^c(n) \quad \text{and} \quad G^{(i)}(t) = \frac{d^i}{dt^i}G(t). \quad (9)$$

The PHD of the IIDC RFS is given as:

$$D^c(\mathbf{x}) = \int f(\{\mathbf{x}\}\cup W)\delta W = \mu_c\cdot q_c(\mathbf{x}) \quad (10)$$

$$\mu_c = \sum_{n=0}^{\infty} n\cdot\pi^c(n) = G^{(1)}(1) \quad (11)$$

where μ_c is the mean cardinality. Its second factorial moment can be calculated to be [14]

$$D^c(\{\mathbf{x}_1,\mathbf{x}_2\}) = G^{(2)}(1)\cdot q_c(\mathbf{x}_1)\cdot q_c(\mathbf{x}_2) \quad (12)$$

B. Multi-Bernoulli RFS

To understand the Multi-Bernoulli RFS let us first consider the Bernoulli RFS. A Bernoulli RFS can either be an empty set with probability $1-r$ or a singleton set $\{\mathbf{x}\}$ with probability r; the singleton element $\mathbf{x}$ when present is drawn from the distribution function $q(\mathbf{x})$. The multitarget probability density of a Bernoulli RFS is given by

$$f^b(X) = \begin{cases} 1-r, & \text{if } X = \phi \\ r\cdot q(\mathbf{x}), & \text{if } X = \{\mathbf{x}\} \\ 0 & \text{if } |X| > 1.\end{cases} \quad (13)$$

The PHD function of the Bernoulli RFS is given by

$$D^b(\mathbf{x}) = \int f(\{\mathbf{x}\}\cup W)\delta W = r\cdot q(\mathbf{x}). \quad (14)$$

The Bernoulli RFS can be used to model a single target. To represent multiple targets, the multi-Bernoulli RFS is suggested [2]. A multi-Bernoulli RFS χ is defined as the union of N independent Bernoulli RFS components as

$$\chi = \chi_1 \cup \chi_2 \cup \dots \cup \chi_N \quad (15)$$

where each of the χ_i is a Bernoulli RFS with parameters given by $\{r_i, q_i(\mathbf{x})\}_{i=1}^N$. The multitarget density of the multi-Bernoulli RFS can be expressed as

$$f^{mb}(X) = \begin{cases} Q_0, & \text{if } X = \phi \\ Q_0 \times \sum_{1\le i_1\ne\dots\ne i_n\le N}\prod_{j=1}^{n}\frac{r_{i_j}q_{i_j}(\mathbf{x}_j)}{1-r_{i_j}}, & \text{if } |X| = n \le N \\ 0 & \text{if } |X| > N.\end{cases}$$

where $Q_0 = \prod_{j=1}^{N}(1-r_j)$. The PHD of the multi-Bernoulli RFS (Ex. 91, Chap. 16, [2]) can be shown to be

$$D^{mb}(\mathbf{x}) = \sum_{i=1}^{N} r_i\cdot q_i(\mathbf{x}). \quad (16)$$

The second factorial moment of the multi-Bernoulli RFS can be calculated to be [11]

$$D^{mb}(\{\mathbf{x}_1,\mathbf{x}_2\}) = \int f(\{\mathbf{x}_1,\mathbf{x}_2\}\cup W)\delta W \quad (17)$$
$$= \sum_{i=1}^{N}\sum_{j=1,j\ne i}^{N} r_i\cdot r_j\cdot q_i(\mathbf{x}_1)\cdot q_j(\mathbf{x}_2) \quad (18)$$
$$= D^{mb}(\mathbf{x}_1)D^{mb}(\mathbf{x}_2) - \sum_{i=1}^{N} r_i^2\cdot q_i(\mathbf{x}_1)\cdot q_i(\mathbf{x}_2). \quad (19)$$

III. Conditional PHD

A. Conditional PHD definition

Consider a RFS χ which is union of two independent random finite sets χ_A and χ_B. Let the PHD function for each of the independent RFS components be denoted by $D^A(\mathbf{x})$ and

$D^B(\mathbf{x})$ respectively. Let the PHD of the RFS χ be denoted $D(\mathbf{x})$. Then we can show that

$$D(\mathbf{x}) = D^A(\mathbf{x}) + D^B(\mathbf{x}) \tag{20}$$

The above result can be easily proved using the properties of probability-generating functionals and the basic rules for functional derivatives (Chap. 11, [2]).

Let $f(W)$ be the probability density of the RFS χ. We now define the quantity conditional PHD corresponding to the RFS component χ_A as follows

$$\mathcal{D}^A(\mathbf{x}) = \int f(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow A)\delta W \tag{21}$$

The conditioning event $(\mathbf{x} \leftarrow A)$ implies that - if $\mathbf{x}$ is a member of the multitarget state, then element $\mathbf{x}$ is generated by the random finite set χ_A. Similarly we define the conditional PHD $\mathcal{D}^B(\mathbf{x})$ corresponding to the RFS component χ_B. Then we have the following equality relation between the conditional PHD and the PHD of the individual components

$$\mathcal{D}^A(\mathbf{x}) = D^A(\mathbf{x}) \quad \text{and} \quad \mathcal{D}^B(\mathbf{x}) = D^B(\mathbf{x}) \tag{22}$$

The derivation of above result is provided in Appendix A. The above result can be extended when the RFS χ is union of more than two independent RFS. Propagating the complete multitarget density for the RFS χ can be difficult in general. The conditional PHD makes it easier to propagate the posterior multitarget density by allowing us to update the PHD for each individual component of the RFS. This is useful in cases where the component PHDs can be easily related to the complete density. We will make use of this fact in deriving the update equations for the multi-Bernoulli filter and the hybrid multi-Bernoulli CPHD filter where we model the multitarget state as union of multiple independent Bernoulli RFS and IIDC RFS.

B. Conditional PHD update

Theorem 1. *If the predicted and posterior conditional PHD corresponding to RFS component χ_A at time $k+1$ are denoted by $\mathcal{D}^A_{k+1|k}(\mathbf{x})$ and $\mathcal{D}^A_{k+1}(\mathbf{x})$ respectively, then the two conditional PHDs can be shown to be related as*

$$\boxed{\mathcal{D}^A_{k+1}(\mathbf{x}) = \mathcal{D}^A_{k+1|k}(\mathbf{x}) \frac{\int h_{\mathbf{z}_{k+1}}(\{\mathbf{x}\} \cup W) \times f^{\bar{A}}_{k+1|k}(W)\,\delta W}{\int h_{\mathbf{z}_{k+1}}(W) \times f_{k+1|k}(W)\,\delta W}}$$

where, $f_{k+1|k}(W)$ is the predicted multitarget distribution and $f^{\bar{A}}_{k+1|k}(W)$ is the multitarget distribution defined as

$$f^{\bar{A}}_{k+1|k}(W) = \frac{f_{k+1|k}(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow A)}{\mathcal{D}^A_{k+1|k}(\mathbf{x})}. \tag{23}$$

The proof of the above theorem is provided in Appendix B.

IV. Multi-Bernoulli filters for superpositional sensor model

In this section we approximate the result of Theorem 1 for the case of superpositional sensor model under Gaussian sensor noise assumption. This approximation leads to a computationally tractable update equation for the conditional PHD. This result is applied for the specific case of union of independent Bernoulli RFS and IIDC RFS in deriving the update equations for the multi-Bernoulli and hybrid multi-Bernoulli CPHD filters in the later subsections.

A. Approximate conditional PHD update for superpositional sensor model with Gaussian sensor noise

Using the Gaussian sensor noise assumption and the superpositional likelihood model from equations (1) and (2) and applying them to the result of Theorem 1, we have:

$$\mathcal{D}^A_{k+1}(\mathbf{x}) = \mathcal{D}^A_{k+1|k}(\mathbf{x}) \times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - g(\mathbf{x}) - r(W)) \times f^{\bar{A}}_{k+1|k}(W)\,\delta W}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - r(W)) \times f_{k+1|k}(W)\,\delta W}$$

Apply the transformation $\mathbf{y}^{\bar{A}} = r(W)$ in the numerator and $\mathbf{y} = r(W)$ in the denominator. Using the formula for change of variables for set integrals [14] we have,

$$\mathcal{D}^A_{k+1}(\mathbf{x}) = \mathcal{D}^A_{k+1|k}(\mathbf{x}) \times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mathbf{y}^{\bar{A}}) \times Q^{\bar{A}}_{k+1|k}(\mathbf{y}^{\bar{A}})\,d\mathbf{y}^{\bar{A}}}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - \mathbf{y}) \times Q_{k+1|k}(\mathbf{y})\,d\mathbf{y}} \tag{24}$$

where $Q_{k+1|k}(\mathbf{y})$ and $Q^{\bar{A}}_{k+1|k}(\mathbf{y}^{\bar{A}})$ are the probability distributions of the random vectors $\mathbf{y}$ and $\mathbf{y}^{\bar{A}}$ respectively. Using the Gaussian approximation for these densities, $Q_{k+1|k}(\mathbf{y}) \approx \mathcal{N}_{\Sigma_{k+1}}(\mathbf{y} - \mu_{k+1})$ and $Q^{\bar{A}}_{k+1|k}(\mathbf{y}^{\bar{A}}) \approx \mathcal{N}_{\Sigma^{\bar{A}}_{k+1}}(\mathbf{y}^{\bar{A}} - \mu^{\bar{A}}_{k+1})$,

$$\mathcal{D}^A_{k+1}(\mathbf{x}) \approx \mathcal{D}^A_{k+1|k}(\mathbf{x}) \times \frac{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mathbf{y}^{\bar{A}}) \times \mathcal{N}_{\Sigma^{\bar{A}}_{k+1}}(\mathbf{y}^{\bar{A}} - \mu^{\bar{A}}_{k+1})\,d\mathbf{y}^{\bar{A}}}{\int \mathcal{N}_{\Sigma_r}(\mathbf{z}_{k+1} - \mathbf{y}) \times \mathcal{N}_{\Sigma_{k+1}}(\mathbf{y} - \mu_{k+1})\,d\mathbf{y}}$$

The above equation can be simplified using the following calculus result, $\int \mathcal{N}_{\Sigma_1}(a - \mathbf{y}) \times \mathcal{N}_{\Sigma_2}(\mathbf{y} - b)\,d\mathbf{y} = \mathcal{N}_{\Sigma_1+\Sigma_2}(a-b)$. Thus the approximate conditional PHD update equation for superpositional sensor model is

$$\boxed{\mathcal{D}^A_{k+1}(\mathbf{x}) \approx \mathcal{D}^A_{k+1|k}(\mathbf{x}) \frac{\mathcal{N}_{\Sigma_r + \Sigma^{\bar{A}}_{k+1}}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mu^{\bar{A}}_{k+1})}{\mathcal{N}_{\Sigma_r + \Sigma_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})}} \tag{25}$$

where μ_{k+1} and Σ_{k+1} are the mean and covariance matrix of the distribution $Q_{k+1|k}(\mathbf{y})$ and $\mu^{\bar{A}}_{k+1}$ and $\Sigma^{\bar{A}}_{k+1}$ are the mean and covariance matrix of the distribution $Q^{\bar{A}}_{k+1|k}(\mathbf{y}^{\bar{A}})$. These mean and covariance matrix parameters can be found using the quadratic version of Campbell's theorem [14], [15]. By modelling the unknown multi-target state as union of independent random finite sets, different tracking filters can be derived whose update equations are special cases of equation (25). Examples of two such filters are discussed in the next two subsections.

B. Multi-Bernoulli filter

The multi-Bernoulli filter models the multi-target state as the union of multiple independent Bernoulli random finite sets. The scalar existence probability and the state density parameters are propagated over time. The propagation is done in two stages, prediction and update. The single target motion model is used for propogation in the prediction step. The motion model accounts for the survival of existing targets from previous time step to current time step and for the birth of new targets. We do not consider target spawning in this paper. The most recent observation along with the likelihood model relating the observations to the multi-target state is used in the update step to propagate the Bernoulli parameters. The following subsections discuss the prediction and update step in detail for the multi-Bernoulli filter.

1) Prediction step: The multi-Bernoulli prediction equations are derived in [2], [17]. Since the superpositional observation model does not play a role in the prediction step, the multi-Bernoulli prediction equations remain the same. We briefly review these equations in this section.

Let the existence probability and state density parameters of the N_k targets at time k be $\{r_{k,i}, q_{k,i}(\mathbf{x})\}_{i=1}^{N_k}$. At time $k+1$ let there be $N_{k+1|k}$ predicted targets with parameters $r_i = r_{k+1|k,i}$ and $q_i(\mathbf{x}) = q_{k+1|k,i}(\mathbf{x})$. Additionally, the predicted multi-Bernoulli RFS parameters can be expressed as

$$\{r_i, q_i(\mathbf{x})\}_{i=1}^{N_{k+1|k}} = \{r_i^P, q_i^P(\mathbf{x})\}_{i=1}^{N_k} \cup \{r_i^B, q_i^B(\mathbf{x})\}_{i=N_k+1}^{N_{k+1|k}}$$

where $\{r_i^P, q_i^P(\mathbf{x})\}_{i=1}^{N_k}$ are the parameters of targets propagated from the previous time step and $\{r_i^B, q_i^B(\mathbf{x})\}_{i=N_k+1}^{N_{k+1|k}}$ are the parameters of newly born targets. The relation between the the predicted target parameters at time $k+1$ and the posterior target parameters at time k is

$$r_i^P = r_{k,i} \times \langle q_{k,i}, p_s \rangle, \tag{26}$$

$$q_i^P(\mathbf{x}) = \frac{\langle t_{k+1|k}(\mathbf{x}|\cdot), q_{k,i} p_s \rangle}{\langle q_{k,i}, p_s \rangle} \tag{27}$$

where, $p_s(\mathbf{x})$ is the target survival probability, $t_{k+1|k}(\mathbf{x}|\cdot)$ is the Markov transition kernel and $\langle a, b \rangle$ is the scalar product defined as $\langle a, b \rangle = \int a(\mathbf{x}) b(\mathbf{x}) d\mathbf{x}$. Since the parameters $\{r_i^B, q_i^B(\mathbf{x})\}_{i=N_k+1}^{N_{k+1|k}}$ are used to model the new targets arriving at time $k+1$, they are initialized using the target birth model.

2) Update step: We assume that the posterior multi-target density also has the multi-Bernoulli form. For the purpose of presenting a simpler and clearer derivation, we assume that no new targets are added in the update step and hence $N_{k+1} = N_{k+1|k}$. Since the collective conditional PHD of all the Bernoulli components can completely specify the posterior multi-Bernoulli density, the update step consists of updating the conditional PHD for each of the $i = 1, 2, \ldots, N_{k+1}$ Bernoulli components. Let $\{r_i', q_i'(\mathbf{x})\}_{i=1}^{N_{k+1}}$ denote the parameter set of the posterior multi-Bernoulli density at time $k+1$. Combining the results from equations (14), (22) and (25), the approximate conditional PHD update is given by

$$\boxed{r_i' \cdot q_i'(\mathbf{x}) \approx r_i \cdot q_i(\mathbf{x}) \frac{\mathcal{N}_{\Sigma_r + \Sigma_{k+1}^{\bar{i}}}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mu_{k+1}^{\bar{i}})}{\mathcal{N}_{\Sigma_r + \Sigma_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})}} \tag{28}$$

where,

$$\mu_{k+1} = \sum_{i=1}^{N_{k+1|k}} r_i \cdot s_i \tag{29}$$

$$\Sigma_{k+1} = \sum_{i=1}^{N_{k+1|k}} (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) \tag{30}$$

$$s_i = \langle q_i, g \rangle, \quad v_i = \langle q_i, g g^T \rangle \quad \text{and}$$

$$\mu_{k+1}^{\bar{i}} = \mu_{k+1} - r_i \cdot s_i \tag{31}$$

$$\Sigma_{k+1}^{\bar{i}} = \Sigma_{k+1} - (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) \tag{32}$$

The expressions for the parameters above are derived in Appendix C.

C. Hybrid multi-Bernoulli CPHD filter

The multi-Bernoulli RFS modelling of the multitarget state allows us to model each of the targets individually and update its state information. Though this is an improvement over the IIDC RFS modeling of the multitarget state which utilizes only one state density function to model all the targets, it does not have a robust cardinality representation. Also, since the number of targets is changing over time, we need to add multiple Bernoulli components at each time step to account for the new born targets. Processing a large number of Bernoulli components at each time step is not computationally efficient. To address these drawbacks we propose to use a hybrid approach where the existing targets are modeled using a multi-Bernoulli RFS and the new born targets are modeled using the IIDC RFS.

The hybrid multi-Bernoulli CPHD filter uses the following modeling scheme. The final posterior distribution from the previous time step is modeled as a multi-Bernoulli RFS. In the prediction step, the multi-Bernoulli component is propagated following the motion model of surviving targets whereas to account for new born targets an IIDC RFS component is initialized. The IIDC component is independent of the multi-Bernoulli component. Thus the predicted distribution corresponds to the union of independent IIDC RFS and multi-Bernoulli RFS which is completely represented by the conditional PHD of the Bernoulli components, the conditional PHD of the IIDC component and the cardinality distribution of the IIDC component. The update step propagates all of these quantities using the Bayes' rule. Hence the obtained posterior comprises of the union of an IIDC component and a multi-Bernoulli component. Since individual targets are better represented using Bernoulli RFSs, the updated IIDC component is approximated using multiple Bernoulli components. Thus the final posterior distribution is approximately modeled using a multi-Bernoulli RFS.

1) Prediction step: Let the parameters of the posterior Bernoulli components at time step k be denoted $\{r_{k,i}, q_{k,i}(\mathbf{x})\}_{i=1}^{N_k}$ as before. The Bernoulli parameters at the end of the prediction step are $\{r_i, q_i(\mathbf{x})\}_{i=1}^{N_k}$ and are given by the equations (26) and (27). Let $\pi_{k+1|k}^c(n)$ and $q_c(\mathbf{x}) = q_{c,k+1|k}(\mathbf{x})$ be the predicted cardinality distribution and the predicted density function at time $k+1$. Their exact forms depend on the specific target birth model used. Let μ_c denote the expected cardinality of the predicted IIDC RFS component. Note that no new Bernoulli components are added in the prediction step to account for birth of new targets.

2) Update step: The hybrid multi-Bernoulli filter assumes that the posterior multitarget state is the union of independent IIDC and multi-Benoulli RFSs. Hence the update step consists of updating the conditional PHD for each of the Bernoulli components, conditional PHD of the IIDC component and the cardinality distribution of the IIDC component. Let the parameters of the multi-Bernoulli and IIDC RFSs be denoted by $\{r'_i, q'_i(\mathbf{x})\}_{i=1}^{N_k}$ and $\{q'_c(\mathbf{x}), \pi^c_{k+1}(n)\}$ respectively.

From equations (14), (22) and (25), the conditional PHD update of the i^{th} Bernoulli component is given by

$$\boxed{r'_i \cdot q'_i(\mathbf{x}) \approx r_i \cdot q_i(\mathbf{x}) \frac{\mathcal{N}_{\Sigma_r + \Sigma^{\bar{i}}_{k+1}}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mu^{\bar{i}}_{k+1})}{\mathcal{N}_{\Sigma_r + \Sigma_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})}} \quad (33)$$

where,

$$\mu_{k+1} = \sum_{i=1}^{N_k} r_i \cdot s_i + \mu_c \cdot s_c \quad (34)$$

$$\Sigma_{k+1} = \sum_{i=1}^{N_k} (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) + \mu_c \cdot v_c - \left(\mu_c^2 - G^{(2)}(1)\right) \cdot s_c s_c^T \quad (35)$$

$$\mu^{\bar{i}}_{k+1} = \mu_{k+1} - r_i \cdot s_i \quad (36)$$

$$\Sigma^{\bar{i}}_{k+1} = \Sigma_{k+1} - (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) \quad (37)$$

$$s_i = \langle q_i, g \rangle, \quad v_i = \langle q_i, g g^T \rangle, \quad (38)$$

$$s_c = \langle q_c, g \rangle, \quad v_c = \langle q_c, g g^T \rangle \quad (39)$$

The equations for the parameters μ_{k+1}, Σ_{k+1}, $\mu^{\bar{i}}_{k+1}$ and $\Sigma^{\bar{i}}_{k+1}$ above are derived in Appendix C using the quadratic version of Campbell's theorem.

Using equations (10), (22) and (25), the conditional PHD update for the IIDC RFS component is

$$\boxed{\mu'_c \cdot q'_c(\mathbf{x}) \approx \mu_c \cdot q_c(\mathbf{x}) \frac{\mathcal{N}_{\Sigma_r + \Sigma^{\bar{c}}_{k+1}}(\mathbf{z}_{k+1} - g(\mathbf{x}) - \mu^{\bar{c}}_{k+1})}{\mathcal{N}_{\Sigma_r + \Sigma_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})}} \quad (40)$$

where,

$$\mu^{\bar{c}}_{k+1} = \sum_{j=1}^{N_k} r_j \cdot s_j + \frac{G^{(2)}(1)}{\mu_c} \cdot s_c, \quad (41)$$

$$\Sigma^{\bar{c}}_{k+1} = \sum_{j=1}^{N_k} (r_j \cdot v_j - r_j^2 \cdot s_j s_j^T) + \frac{G^{(2)}(1)}{\mu_c} v_c - \left(\frac{G^{(2)}(1)^2}{\mu_c^2} - \frac{G^{(3)}(1)}{\mu_c} \right) s_c s_c^T \quad (42)$$

The parameters μ_{k+1} and Σ_{k+1} are as given in equations (34) and (35) respectively. The derivation of the parameters $\mu^{\bar{c}}_{k+1}$ and $\Sigma^{\bar{c}}_{k+1}$ is given in Appendix C.

The main advantage of the IIDC component in the hybrid filter is that we can make use of the accurate cardinality estimation of the CPHD filter. The update equation for the cardinality distribution of the IIDC RFS component can be shown to be

$$\boxed{\pi^c_{k+1}(m_1) \approx \pi^c_{k+1|k}(m_1) \frac{\mathcal{N}_{\Sigma_r + \Sigma^{c,m_1}_{k+1}}(\mathbf{z}_{k+1} - \mu^{c,m_1}_{k+1})}{\mathcal{N}_{\Sigma_r + \Sigma_{k+1}}(\mathbf{z}_{k+1} - \mu_{k+1})}} \quad (43)$$

where,

$$\mu^{c,m_1}_{k+1} = \sum_{i=1}^{N_{k+1|k}} r_i \cdot s_i + m_1 \cdot s_c$$

$$\Sigma^{c,m_1}_{k+1} = \sum_{i=1}^{N_{k+1|k}} (r_i \cdot v_i - r_i^2 \cdot s_i s_i^T) + m_1(v_c - s_c s_c^T)$$

The parameters μ_{k+1} and Σ_{k+1} are as given in equations (34) and (35) respectively. The derivation of the above cardinality update equation is given in Appendix D. Note that the multi-Bernoulli filter can be treated as a special case of the hybrid multi-Bernoulli CPHD filter. Indeed we get back the multi-Bernoulli filter update equations if we set the IIDC component to be empty set in all the above equations.

V. Auxiliary particle filter implementation

We study the proposed filters using the Monte Carlo approach. Approximate update equations have been derived in this paper but they do not lead to a fully analytically tractable filter. Hence we develop particle filter based implementations of the filters. The basic particle filter approach does not give a stable implementation because of the multiple approximations employed to derive the filter equations. We propose auxiliary particle filter implementation of the multi-Bernoulli filter and the hybrid multi-Bernoulli filter.

A. Multi-Bernoulli filter

The conditional PHD corresponding to each Bernoulli component is approximated using a set of weighted particles as follows

$$r_i \cdot q_i(\mathbf{x}) \approx \sum_{j=1}^{N_p} w_i^j \delta(\mathbf{x} - \mathbf{x}_i^j); \quad \sum_{j=1}^{N_p} w_i^j = r_i \quad (44)$$

Thus we use one particle filter for each Bernoulli component. The pseudo code for the auxiliary particle filter implementation of the multi-Bernoulli filter is provided in Algorithm 1. New components are initialized following the target birth model. In the prediction step particles are propagated according to the motion model. The update is performed in two stages so that the new born targets are accurately identified. The conditional PHD update step is realized by performing update of particle weights using equation (28). The required statistics are obtained through numerical approximation using weighted particles, for example,

$$s_i = \langle q_i, g \rangle \approx \sum_{j=1}^{N_p} w_i^j g(\mathbf{x}_i^j)$$

$$v_i = \langle q_i, g g^T \rangle \approx \sum_{j=1}^{N_p} w_i^j g(\mathbf{x}_i^j) g(\mathbf{x}_i^j)^T$$

For the new components the auxiliary filtering step accounts for the fact that we are sampling from a different different distribution than the birth model. Pruning of Bernoulli components is performed after the second update stage so as to eliminate targets with low probability of existence ($< r_0$). The new components created can sometimes correspond to existing targets. Hence gating is performed so that new targets starting within close vicinity of existing targets are pruned.

B. Hybrid multi-Bernoulli CPHD filter

The pseudo code for the auxiliary particle filter implementation of the hybrid multi-Bernoulli CPHD filter is provided

for $k = 1$ to T **do**
  Prediction
  Propagate existing components $\{r_i, q_i(\mathbf{x})\}_{i=1}^{N_{k-1}}$ using equations (26) and (27)
  Add new Bernoulli components - $\{r_i, q_i(\mathbf{x})\}_{i=N_{k-1}+1}^{N_{k|k-1}}$
  First Run
  Calculate statistics - μ_k, Σ_k
  Update conditional PHD for $i = N_{k-1}+1 \dots N_{k|k-1}$ using equation (28)
  Auxiliary filtering step for $i = N_{k-1}+1 \dots N_{k|k-1}$
  Update statistics - μ_k, Σ_k
  Update conditional PHD for $i = 1 \dots N_{k|k-1}$ using equation (28)
  Auxiliary filtering step for $i = 1 \dots N_{k-1}$
  Second Run
  Update statistics - μ_k, Σ_k
  Update conditional PHD for $i = 1 \dots N_{k|k-1}$ using equation (28)
  Resample conditional PHD for $i = 1 \dots N_{k|k-1}$
  Track management
  Prune Bernoulli tracks with $r_i < r_0$
  Gate new components to check for duplicity
end for

Fig. 1. Pseudo-code for auxiliary particle filter implementation of multi-Bernoulli filter.

in Algorithm 2. One particle filter is used for each Bernoulli component and one particle filter is used to approximate the conditional PHD of the IIDC component. The IIDC conditional PHD is initialized using the target birth model. The IIDC cardinality distribution is represented using a finite dimensional vector whose elements sum to one. After the second update stage, an estimate of the new born targets $N^c_{k|k}$ is obtained from the IIDC cardinality distribution. The set of particles representing IIDC conditional PHD is then partitioned into $N^c_{k|k}$ clusters and each cluster is used to initialize a new Bernoulli component. Gating is performed to check for duplicity between the new and existing components. The pruned Bernoulli components are used for initialization of IIDC conditional PHD along with the target birth model in the next time step. This can be helpful in cases of low detection probability or high noise.

VI. Numerical simulations

The above filters can be used for state estimation in various real life applications whenever the sensor observations have the superpositional form described in Section I-C. In this section we demonstrate successful application of these filters to the problem of multitarget tracking. The measurement modality of radio frequency sensors are studied using the simulations. A network of RF sensors is used to monitor a region of interest where multiple targets can be simultaneously present and can appear or disappear over time.

for $k = 1$ to T **do**
  Prediction
  Propagate existing components $\{r_i, q_i(\mathbf{x})\}_{i=1}^{N_{k-1}}$ using equations (26) and (27)
  Initialize IIDC component using $q_c(\mathbf{x})$ and $\pi^c_{k|k-1}(n)$
  First Run
  Calculate statistics - μ_k, Σ_k
  Update IIDC conditional PHD using equation (40)
  Auxiliary filtering step for IIDC conditional PHD
  Update cardinality distribution using equation (43)
  Update statistics - μ_k, Σ_k
  Update Bernoulli conditional PHD for $i = 1 \dots N_{k-1}$ using equation (33)
  Auxiliary filtering step for $i = 1 \dots N_{k-1}$
  Second Run
  Update statistics - μ_k, Σ_k
  Update Bernoulli conditional PHD for $i = 1 \dots N_{k-1}$ using equation (33)
  Update IIDC conditional PHD using equation (40)
  Update cardinality distribution using equation (43)
  Resample conditional PHD for Bernoulli and IIDC components
  Approximation and track management
  Prune Bernoulli tracks with $r_i < r_0$
  Estimate IIDC cardinality $N^c_{k|k}$ = MAP $\{\pi^c_k(n)\}$
  Cluster IIDC conditional PHD into $N^c_{k|k}$ components
  Initialize new Bernoulli components using clusters
  Gate new components to check for duplicity
end for

Fig. 2. Pseudo-code for auxiliary particle filter implementation of hybrid multi-Bernoulli CPHD filter.

A. Target dynamics

We assume that for each target its dynamics are independent of the other targets and their dynamics. Specifically, motion of each target when present within the monitoring region is governed by the following approximately constant velocity model [25]:

$$\mathbf{x}_{k+1,i} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_{k,i} + \begin{bmatrix} \frac{T^2}{2} & 0 \\ 0 & \frac{T^2}{2} \\ T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad (45)$$

where T is the sampling period and u_x, u_y are zero-mean Gaussian white noise with respective variance $\sigma^2_{u_x}$ and $\sigma^2_{u_y}$. In this model, the state of each object i at time k, $\mathbf{x}_{k,i}$, is represented by a four-dimensional vector: position on the x-axis and y-axis, velocity on the x-axis and y-axis.

Figure 3(a) shows the target tracks we use for the simulations. The black cross (x) indicates the starting location of the target. The variation of number of targets over time is shown in Figure 3(b). Target labelled number 7 and 8 in the Figure 3(a) appear within the monitoring region at time steps 9 and 17 respectively. Target 8 disappears from the monitoring region at time step 24. The mimimum distance between any two targets as a function of time is shown in Figure 3(c). As can be seen, some of the targets approach very close to each other (less than $1m$) and on multiple occasions. The targets in the above scenario evolve according to the linear Gaussian dynamics given in Equation (45) with a time step of duration $T = 0.25s$ and the noise variance parameters of $\sigma^2_{u_x} = \sigma^2_{u_y} = 0.35$. We simulate 35 time steps of target motion for a total of $35 \times 0.25 = 8.75s$.

B. Error metric

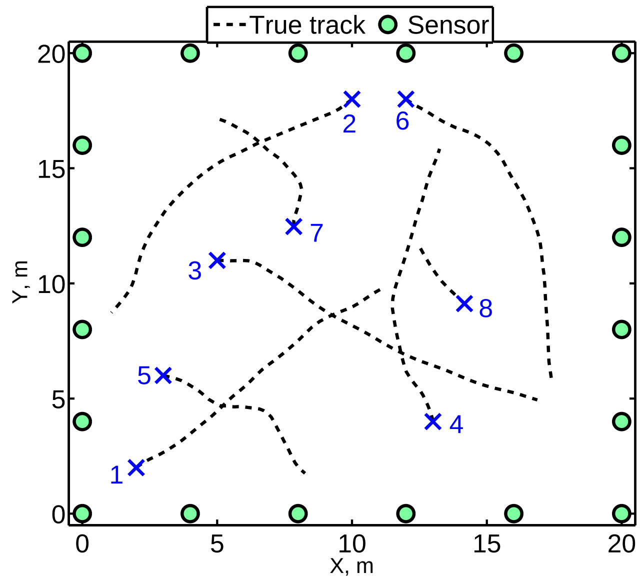
We will compare the multi-Bernoulli filters with the CPHD filter. To compare them, we need to find the error between the filter estimates and the true multitarget state. Since we have to compare sets, possibly with different cardinality, we use the optimal subpattern assignment (OSPA) metric [26]. The OSPA metric penalises errors in target location estimate as well as error in estimating the number of targets using the cardinality penalty factor c. When there are n targets and we estimate m targets then for $m \leq n$ the OSPA metric is defined as

$$d_p^{(c)}(X,Y) = \left(\frac{1}{n} \min_{\pi \in \Pi} \sum_{i=1}^{m} d^{(c)}(x_i, y_{\pi(i)})^p + c^p(n-m) \right)^{1/p} \tag{46}$$

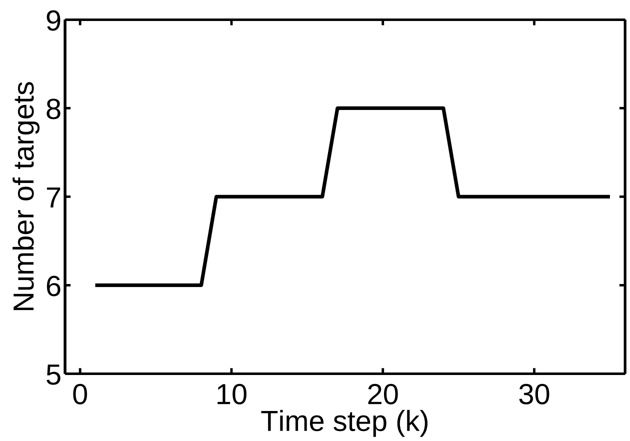
where Π is the set of possible permutations of $\{1, 2, \ldots, n\}$, $d(x,y)$ is the Euclidean distance between x and y and $d^{(c)}(x,y) = \min\{d(x,y), c\}$. $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ are arbitrary sets and p is a fixed parameter. We use the value $p = 2$. When $m > n$, we calculate $d_p^{(c)}(Y,X)$. The OSPA metric finds the best permutation of the larger set which minimizes its distance from the smaller set and assigns a fixed penalty for each cardinality error. We compute the average OSPA error by performing multiple executions of the filter each with a different random initialization.

C. RF tomography

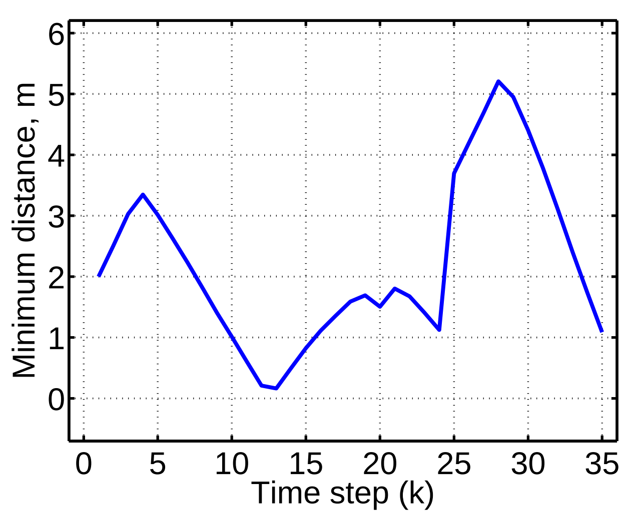
A typical deployment of RF sensors for tomography application is shown in Figure 3(a). The measurements are the received signal strength (RSS) recordings for each sensor pair. The RF sensors communicate among each other but not with the targets. A network of N_s sensors forms a total of $n_{\mathbf{z}} = N_s(N_s - 1)/2$ unique sensor pairs (bidirectional links) generating $n_{\mathbf{z}}$ measurements in every time step. The background RSS values are recorded initially when the monitoring region is empty. The objective of RF tomography is to use the measured deviations from these background RSS values to track moving targets.



(a) Target tracks and sensor locations



(b) Cardinality



(c) Minimum distance between targets

Fig. 3. Top: Target tracks used in the simulation. The black cross (x) indicates the initial location of the target and green circles represent sensors. Middle: Variation of number of targets. The 7^{th} and the 8^{th} target appear at time steps 9 and 17 respectively. The 8^{th} target disappears at time step 24. Bottom: The minimum distance between any two targets as a function of time.

The empirical modeling of RF link measurements are studied in [27]–[29]. The j^{th} link measurement z_k^j at time step k can be modeled as:

$$z_k^j = r^j(X_k) + \mathbf{v}_k^j = \sum_{\mathbf{x}\in X_k} g^j(\mathbf{x}) + \mathbf{v}_k^j$$
$$\text{where, } g^j(\mathbf{x}) = \phi \exp\left(-\frac{\lambda_j(\mathbf{x})}{\sigma_\lambda}\right). \quad (47)$$

Here $\lambda_j(\mathbf{x})$ is an elliptical distance measure between a target located at $\mathbf{x}$ and link j (see [27] for more details); ϕ and σ_λ are fixed parameters based on physical properties of the sensors; $\mathbf{v}_k^j$ is the zero-mean Gaussian sensor noise. The RF tomography measurement equation has a superpositional form as can be seen by comparing equation (47) with equation (1) in Section I-C.

The radio frequency tomography approach to target tracking has recently become popular [8], [27], [30]. The RF tomography measurement model we simulate here is based on the work by Li et al. where it is used for single target tracking [27] and the work by Nannuru et al. where it is used for multitarget tracking [28], [29] from field deployments of RF sensor networks. The model has also been used for multitarget tracking in computer simulations [13], [15].

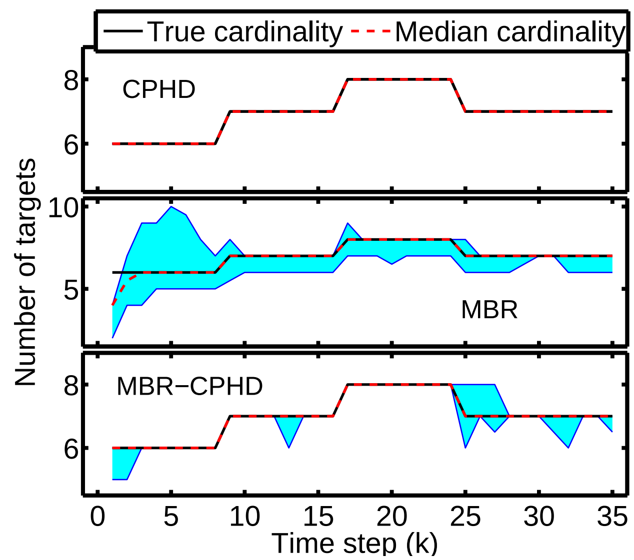
We simulate the above RF sensor network with $N_s = 20$ sensor nodes distributed uniformly on the periphery of the $20m \times 20m$ square region as shown in Figure 3(a). This gives rise to a total of $n_{\mathbf{z}} = 190$ unique bidirectional links. The observation model parameters are $\phi = 5$ and $\sigma_\lambda = 0.4$. The measurement noise variance is $\Sigma_r = \sigma_r^2 \cdot I_{n_{\mathbf{z}}}$ where $\sigma_r^2 = 0.25$ and $I_{n_{\mathbf{z}}}$ is the $n_{\mathbf{z}} \times n_{\mathbf{z}}$ identity matrix.

The average OSPA error metrics are reported by running 100 Monte Carlo simulations (by changing initial seed) of each algorithm for each of the experiments. The target tracks are fixed for the different random initializations. The mean and standard deviation (SD) results are summarized in the Table I. We ignore the first 5 time steps while calculating the average error to allow the filter estimates to stabilize. The hybrid MBR-CPHD filter has the lowest average OSPA error among all the filters. Run 1000 simulations ?

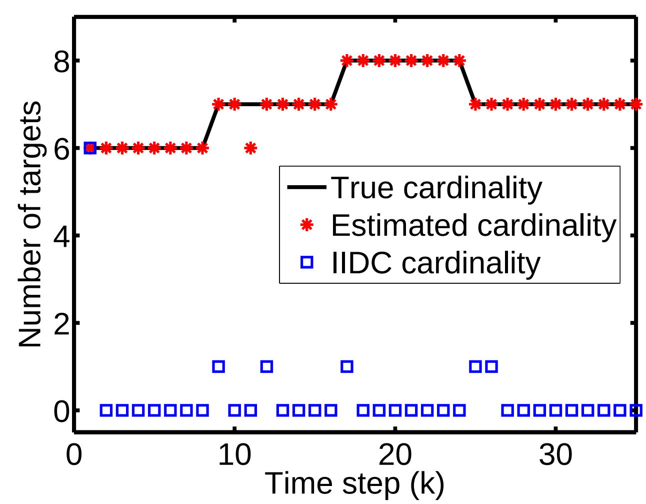
TABLE I
RADIO-FREQUENCY SENSORS: MEAN AND STANDARD DEVIATION OF THE OSPA ERROR FOR DIFFERENT VALUES OF CARDINALITY PENALTY FACTOR c.

Algorithm	OSPA error (Mean ± SD)		
	$c = 0.5$	$c = 1$	$c = 1.5$
CPHD	0.17 ± 0.01	0.17 ± 0.01	0.18 ± 0.02
MBR	0.18 ± 0.03	0.26 ± 0.08	0.33 ± 0.13
MBR-CPHD	0.14 ± 0.01	0.15 ± 0.02	0.17 ± 0.03

The median cardinality estimates (averaged over the 100 Monte Carlo simulations) and its percentiles for the different algorithms are shown in Figure 4(a). The multi-Bernoulli (MBR) filter has low initial cardinality estimates because only a maximum of 4 new Bernoulli components are added at each time step. The CPHD filter has the most accurate cardinality estimates. The hybrid MBR-CPHD filter can occasionally make error in estimating the number of targets but is subsequently corrected by input from the CPHD component at the next time step. An example of this is illustrated in Figure 4(b) which shows the cardinality estimates of the IIDC component and the final cardinality estimate of the hybrid MBR-CPHD filter. At time step $k = 11$ one of the Bernoulli component gets terminated but is revived in the next time step ($k = 12$) using the IIDC component. Note that the IIDC component also correctly identifies arrival of new targets at time steps $k = 9$ and $k = 17$.



(a) Median cardinality



(b) Cardinality for a single run

Fig. 4. Top: Median cardinality and its 5-95 percentiles (cyan colored region) as a function of time. Bottom: The cardinality estimates for a single run of the hybrid MBR-CPHD filter.

We study the variation of OSPA error with time using the box and whisker plots. Figure 5 plots the box and whisker diagram for the OSPA error. The plots are generated using the OSPA error with a cardinality penalty factor of $c = 0.5$. The hybrid MBR-CPHD filter has slightly lower median error than the CPHD filter which has much lower median error than the MBR filter. The median error for the MBR filter

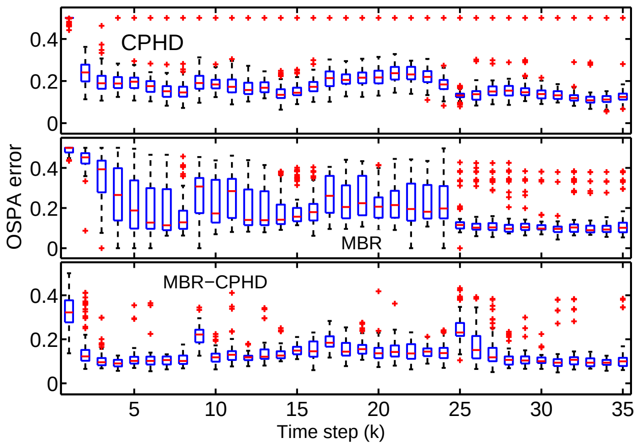


Fig. 5. Box and whisker plot of the OSPA error as a function of time. Boxes indicate 25-75 interquartile range; whiskers extend 1.5 times the range and '+' symbols indicate outliers lying beyond the whiskers.

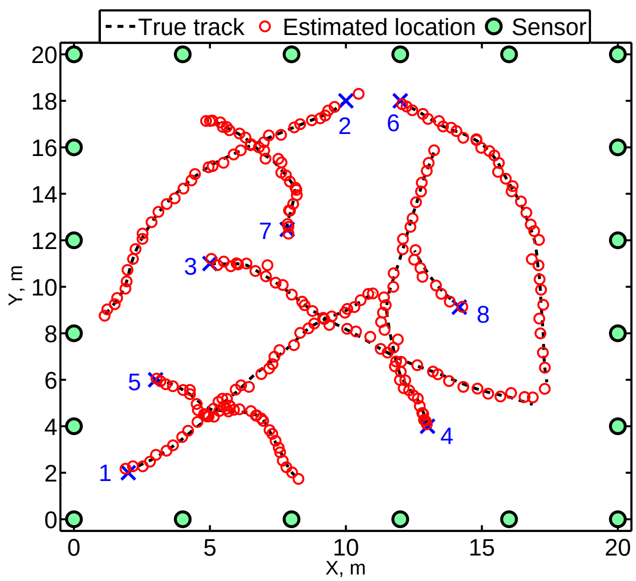


Fig. 6. True target tracks and estimated target locations obtained using the hybrid MBR-CPHD filter.

is high because it is prone to frequent cardinality errors. The extreme outliers of the CPHD filter correspond to one of the failed simulations. The hybrid MRB-CPHD filter can make cardinality errors which causes higher number of outliers which are relatively farther from the median. Add more discussion on results - error tables, box and whisker plots. An example target location estimates obtained using the hybrid MBR-CPHD filter are shown in Figure 6.

Table II compares the computational time required for the different algorithms. The hybrid MBR-CPHD filter is faster among all the filters. Individually, the CPHD filter has higher computational requirement because of the costly clustering step required at each time step and the MBR filter has higher computational requirement because of the multiple additional particle filters employed to account for new target arrivals. The hybrid MBR-CPHD filter saves computation by initiating new particle filters only if the IIDC component indicates arrival of new targets and the costly clustering step is required only when multiple new targets arrive within the monitoring region.

TABLE II
RADIO-FREQUENCY SENSORS: COMPUTATIONAL TIME REQUIRED IN SECONDS.

Algorithm	Time (Mean ± SD, seconds)
CPHD	45.61 ± 2.11
MBR	35.73 ± 0.95
MBR-CPHD	28.11 ± 0.52

APPENDIX A
CONDITIONAL PHD FOR UNION OF INDEPENDENT RANDOM FINITE SETS

Let $f^A(W)$ and $f^B(W)$ be the multitarget probability densities of two independent random finite sets χ_A and χ_B respectively. Let $f(W)$ be the multitarget probability density of $\chi = \chi_A \cup \chi_B$. Then we have the following relation between the densities (Sec. 11.5.3, Chap. 11, [2])

$$f(W) = \sum_{Y \subset W} f^A(Y) f^B(W - Y) \tag{48}$$

where the summation is over all subsets Y of W. The set $(W - Y)$ is the difference set which includes elements from W not present in Y. The conditional PHD corresponding to RFS component χ_A is

$$\begin{aligned}
\mathcal{D}^A(\mathbf{x}) &= \int f(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow A) \delta W \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{|W|=n} f(\{\mathbf{x}\} \cup W | \mathbf{x} \leftarrow A) dW \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{|W|=n} \left(\sum_{Y \subset W} f^A(\{\mathbf{x}\} \cup Y) f^B(W - Y) \right) dW \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{Y \subset W} \int_{|W|=n} f^A(\{\mathbf{x}\} \cup Y) f^B(W - Y) dW \right)
\end{aligned}$$

In the above expression let $W - Y = Z$ so that we can write $W = Y \cup Z$ where $Y \cap Z = \phi$. Hence we can seperate the integration for the different variables as $dW = dY dZ$. Also let $|Y| = m$, $|Z| = n - m = l$. Thus we can write

$$\begin{aligned}
&\sum_{Y \subset W} \int_{|W|=n} f^A(\{\mathbf{x}\} \cup Y) f^B(W - Y) dW \\
&= \sum_{Y \cup Z = W, Y \cap Z = \phi} \int_{|W|=n} f^A(\{\mathbf{x}\} \cup Y) f^B(Z) dY dZ \\
&= \sum_{m=0}^{n} \binom{n}{m} \left(\int_{|Y|=m} f^A(\{\mathbf{x}\} \cup Y) dY \right) \left(\int_{|Z|=l} f^B(Z) dZ \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathcal{D}^A(\mathbf{x}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \left(\int_{|Y|=m} f^A(\{\mathbf{x}\} \cup Y) dY \right) \left(\int_{|Z|=l} f^B(Z) dZ \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{n!} \binom{n}{m} \left(\int_{|Y|=m} f^A(\{\mathbf{x}\} \cup Y) dY \right) \left(\int_{|Z|=l} f^B(Z) dZ \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{1}{n!} \binom{n}{m} \left(\int_{|Y|=m} f^A(\{\mathbf{x}\} \cup Y) dY \right) \left(\int_{|Z|=l} f^B(Z) dZ \right)
\end{aligned}$$

where the last step is rewritten by interchanging the order of the two summations. We also have

$$\frac{1}{n!}\binom{n}{m} = \frac{1}{n!}\frac{n!}{m!(n-m)!} = \frac{1}{m!}\frac{1}{(n-m)!} = \frac{1}{m!}\frac{1}{l!}$$

Continuing from above,

$$\begin{aligned}
&\mathcal{D}^A(\mathbf{x}) \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{m!}\frac{1}{l!}\left(\int_{|Y|=m} f^A(\{\mathbf{x}\}\cup Y)dY\right)\left(\int_{|Z|=l} f^B(Z)dZ\right) \\
&= \left(\sum_{m=0}^{\infty}\frac{1}{m!}\int_{|Y|=m} f^A(\{\mathbf{x}\}\cup Y)dY\right)\left(\sum_{l=0}^{\infty}\frac{1}{l!}\int_{|Z|=l} f^B(Z)dZ\right)
\end{aligned}$$

By the definition of set integral and using equation (3) for the multitarget probability density $f^B(Z)$, the second summation above is equal 1. Hence we have,

$$\begin{aligned}
\mathcal{D}^A(\mathbf{x}) &= \sum_{m=0}^{\infty}\frac{1}{m!}\int_{|Y|=m} f^A(\{\mathbf{x}\}\cup Y)dY \\
&= \int f^A(\{\mathbf{x}\}\cup Y)\delta Y \\
&= D^A(\mathbf{x})
\end{aligned}$$

Appendix B
Proof of Theorem 1

Applying Bayes' rule to definition of conditional PHD,

$$\begin{aligned}
&\mathcal{D}^A_{k+1}(\mathbf{x}) = \int f_{k+1|k+1}(\{\mathbf{x}\}\cup W|\mathbf{x}\leftarrow A)\,\delta W \\
&= \frac{\int h_{\mathbf{z}_{k+1}}(\{\mathbf{x}\}\cup W|\mathbf{x}\leftarrow A) f_{k+1|k}(\{\mathbf{x}\}\cup W|\mathbf{x}\leftarrow A)\,\delta W}{f_{k+1}(\mathbf{z}_{k+1}|Z^{[k]},\mathbf{x}\leftarrow A)} \\
&= \mathcal{D}^A_{k+1|k}(\mathbf{x})\frac{\int h_{\mathbf{z}_{k+1}}(\{\mathbf{x}\}\cup W)\times f^{\bar{A}}_{k+1|k}(W)\,\delta W}{f_{k+1}(\mathbf{z}_{k+1}|Z^{[k]},\mathbf{x}\leftarrow A)}
\end{aligned}$$

where $f^{\bar{A}}_{k+1|k}(W)$ is the multi-target distribution defined as

$$f^{\bar{A}}_{k+1|k}(W) = \frac{f_{k+1|k}(\{\mathbf{x}\}\cup W|\mathbf{x}\leftarrow A)}{\mathcal{D}^A_{k+1|k}(\mathbf{x})}$$

This is a valid distribution which integrates to 1 from the definition of conditional PHD in equation (21). Thus we have

$$\begin{aligned}
\mathcal{D}^A_{k+1}(\mathbf{x}) &= \mathcal{D}^A_{k+1|k}(\mathbf{x}) \\
&\times \frac{\int h_{\mathbf{z}_{k+1}}(\{\mathbf{x}\}\cup W)\times f^{\bar{A}}_{k+1|k}(W)\,\delta W}{\int h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W|Z^{[k]},\mathbf{x}\leftarrow A)\,\delta W}
\end{aligned}$$

To simplify the denominator, we note that

$$\begin{aligned}
&\int h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W|Z^{[k]},\mathbf{x}\leftarrow A)\,\delta W \\
&\qquad = \int h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W)\,\delta W
\end{aligned} \tag{49}$$

This is because the conditional event $(\mathbf{x}\leftarrow A)$ has no effect on the integral. To see this consider the following decomposition of the integral on the left:

$$\begin{aligned}
&\int h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W|Z^{[k]},\mathbf{x}\leftarrow A)\,\delta W \\
&\qquad = \int_{\mathbf{x}\in W} h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W|Z^{[k]},\mathbf{x}\leftarrow A)\,\delta W \\
&\qquad + \int_{\mathbf{x}\notin W} h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W|Z^{[k]})\,\delta W
\end{aligned}$$

But the first integral is zero since $\mathbf{x}\in W$ is a zero probability event. To the second integral we can add the following term, which has zero probability and thus does not affect the evaluation of the integral:

$$\int_{\mathbf{x}\in W} h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W|Z^{[k]})\,\delta W$$

This leads to the expression on the right hand side of equation (49). Hence we have

$$\mathcal{D}^A_{k+1}(\mathbf{x}) = \mathcal{D}^A_{k+1|k}(\mathbf{x})\times\frac{\int h_{\mathbf{z}_{k+1}}(\{\mathbf{x}\}\cup W)\times f^{\bar{A}}_{k+1|k}(W)\,\delta W}{\int h_{\mathbf{z}_{k+1}}(W)\times f_{k+1|k}(W)\,\delta W}$$

Appendix C
Application of Campbell's theorem

Let $f(W)$ be a multitarget density corresponding to some random finite set. Denote the PHD function and the second factorial moment density function of the RFS by $D(\mathbf{x})$ and $D(\{\mathbf{x}_1,\mathbf{x}_2\})$ respectively. Let the random vector $\mathbf{y}$ be a function defined over random sets and have the form $\mathbf{y} = r(W) = \sum_{\mathbf{w}\in W} g(\mathbf{w})$. Then according to quadratic version of Campbell's theorem [14], [15]

$$\mu = E[(\mathbf{y})] = \int g(\mathbf{x})\cdot D(\mathbf{x})d\mathbf{x} \tag{50}$$

$$\begin{aligned}
\Sigma &= E[(\mathbf{y}-\mu)(\mathbf{y}-\mu)^T] \\
&= \int g(\mathbf{x})\cdot g(\mathbf{x})^T\cdot D(\mathbf{x})d\mathbf{x} \\
&+ \int\int g(\mathbf{x}_1)\cdot g(\mathbf{x}_2)^T\cdot\tilde{D}(\{\mathbf{x}_1,\mathbf{x}_2\})d\mathbf{x}_1 d\mathbf{x}_2
\end{aligned} \tag{51}$$

where,

$$\tilde{D}(\{\mathbf{x}_1,\mathbf{x}_2\}) = D(\{\mathbf{x}_1,\mathbf{x}_2\}) - D(\mathbf{x}_1)D(\mathbf{x}_2)$$

Thus the mean and covariance matrix, which represent the first and second order statistics of the random vector $\mathbf{y}$, depend on the PHD function and the second factorial moment density function of the corresponding random set.

If the RFS χ is union of independent RFS χ_A and χ_B, then we have

$$D(\mathbf{x}) = D^A(\mathbf{x}) + D^B(\mathbf{x}) \tag{52}$$

$$\tilde{D}(\{\mathbf{x}_1,\mathbf{x}_2\}) = \tilde{D}^A(\{\mathbf{x}_1,\mathbf{x}_2\}) + \tilde{D}^B(\{\mathbf{x}_1,\mathbf{x}_2\}) \tag{53}$$

The above result can be extended for case of union of more than two independent RFSs.

A. Multi-Bernoulli RFS

For the multi-Bernoulli distribution, using the expression for the PHD and the second-moment density from equations (16) and (19) in the equations (50) and (51) we have

$$\begin{aligned}
\mu_{k+1} &= \int g(\mathbf{x})\cdot\left(\sum_{i=1}^{N_{k+1|k}} r_i\cdot q_i(\mathbf{x})\right)d\mathbf{x} \\
&= \sum_{i=1}^{N_{k+1|k}} r_i\times\int g(\mathbf{x})\cdot q_i(\mathbf{x})d\mathbf{x} \\
&= \sum_{i=1}^{N_{k+1|k}} r_i\cdot s_i,\quad \text{where } s_i = \langle q_i, g\rangle
\end{aligned}$$

$$\Sigma_{k+1} = \int g(\mathbf{x})g(\mathbf{x})^T\left(\sum_{i=1}^{N_{k+1|k}} r_i\cdot q_i(\mathbf{x})\right)d\mathbf{x}$$
$$-\int\int g(\mathbf{x}_1)g(\mathbf{x}_2)^T\left(\sum_{i=1}^{N_{k+1|k}} r_i^2\cdot q_i(\mathbf{x}_1)\cdot q_i(\mathbf{x}_2)\right)d\mathbf{x}_1 d\mathbf{x}_2$$
$$= \sum_{i=1}^{N_{k+1|k}}\left(r_i\cdot v_i - r_i^2\cdot s_i s_i^T\right),\ \text{where } v_i = \langle q_i, gg^T\rangle$$

Combining the multitarget density in equation (23) and the formulas in (48) and (22),

$$\begin{aligned} f^{\bar{A}}(W) &= \frac{f(\{\mathbf{x}\}\cup W|\mathbf{x}\leftarrow A)}{\mathcal{D}^A(\mathbf{x})} \\ &= \frac{\sum_{Y\subset W} f^A(\{\mathbf{x}\}\cup Y)f^B(W-Y)}{D^A(\mathbf{x})} \\ &= \sum_{Y\subset W}\left(\frac{f^A(\{\mathbf{x}\}\cup Y)}{D^A(\mathbf{x})}\right)f^B(W-Y) \\ &= \sum_{Y\subset W}\tilde{f}^A(Y)f^B(W-Y) \end{aligned} \quad (54)$$

where,

$$\tilde{f}^A(Y) = \frac{f^A(\{\mathbf{x}\}\cup Y)}{D^A(\mathbf{x})}.$$

Thus $f^{\bar{A}}(W)$ can be interpreted as the multitarget distribution of the RFS which is union of independent random finite sets with multitarget densities give by $\tilde{f}^A(W)$ and $f^B(W)$. We now consider the multi-Bernoulli RFS χ with $N_{k+1|k}$ parameters to be the union of a Bernoulli RFS (χ_A) with parameters $\{r_i, q_i(\mathbf{x})\}$ and another multi-Bernoulli RFS (χ_B) with remaining parameter set $\{r_j, q_j(\mathbf{x})\}_{j\neq i}$. Using equation (14), the density function $\tilde{f}^A(W)$ corresponds to a RFS which is empty with probability 1 and hence all its moments are zero. By applying the results in equations (52) and (53) to the Campbell's theorem, the parameters $\mu_{k+1}^{\bar{i}}$ and $\Sigma_{k+1}^{\bar{i}}$ are given by

$$\mu_{k+1}^{\bar{i}} = 0 + \sum_{j=1,j\neq i}^{N_{k+1|k}} r_j\cdot s_j = \mu_{k+1} - r_i\cdot s_i$$
$$\Sigma_{k+1}^{\bar{i}} = 0 + \sum_{j=1,j\neq i}^{N_{k+1|k}}\left(r_j\cdot v_j - r_j^2\cdot s_j s_j^T\right)$$
$$= \Sigma_{k+1} - \left(r_i\cdot v_i - r_i^2\cdot s_i s_i^T\right)$$

B. Union of Multi-Bernoulli RFS and IIDC RFS

Let the RFS χ be the union of a multi-Bernoulli RFS with parameters $\{r_i, q_i(\mathbf{x})\}_{i=1}^{N_{k+1|k}}$ and an IIDC RFS with parameters $\{q_c(\mathbf{x}), \pi^c(n)\}$. Using the expression for PHD and second factorial moments from equations (10), (16), (12) and (19) and applying the results in equations (52) and (53) we have

$$\mu_{k+1} = \int g(\mathbf{x})\cdot\left(\sum_{i=1}^{N_{k+1|k}} r_i\cdot q_i(\mathbf{x}) + \mu_c\cdot q_c(\mathbf{x})\right)d\mathbf{x}$$
$$= \sum_{i=1}^{N_{k+1|k}} r_i\cdot s_i + \mu_c\cdot s_c,\quad \text{where } s_c = \langle q_c, g\rangle \quad (55)$$

$$\Sigma_{k+1} = \int g(\mathbf{x})g(\mathbf{x})^T\left(\sum_{i=1}^{N_{k+1|k}} r_i\cdot q_i(\mathbf{x}) + \mu_c\cdot q_c(\mathbf{x})\right)d\mathbf{x}$$
$$-\int\int g(\mathbf{x}_1)g(\mathbf{x}_2)^T\left(\sum_{i=1}^{N_{k+1|k}} r_i^2\cdot q_i(\mathbf{x}_1)\cdot q_i(\mathbf{x}_2)\right)d\mathbf{x}_1 d\mathbf{x}_2$$
$$+\int\int g(\mathbf{x}_1)g(\mathbf{x}_2)^T\left(G^{(2)}(1)-\mu_c^2\right)q_c(\mathbf{x}_1)q_c(\mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2$$
$$= \sum_{i=1}^{N_{k+1|k}}\left(r_i\cdot v_i - r_i^2\cdot s_i s_i^T\right) + \mu_c v_c - \left(\mu_c^2 - G^{(2)}(1)\right)s_c s_c^T$$

where, $v_c = \langle q_c, gg^T\rangle$ (56)

Now consider the extension of the result in equation (54) to union of three independent RFS. Let χ_A be the Bernoulli RFS with parameters $\{r_i, q_i(\mathbf{x})\}$, χ_B be the multi-Bernoulli RFS with parameter set $\{r_j, q_j(\mathbf{x})\}_{j\neq i}$ and χ_C be the IIDC RFS with parameters $\{q_c(\mathbf{x}), \pi^c(n)\}$. Thus using the arguments related to equation (54), we have

$$\mu_{k+1}^{\bar{i}} = \int g(\mathbf{x})\cdot\left(\sum_{j=1,j\neq i}^{N_{k+1|k}} r_j\cdot q_j(\mathbf{x}) + \mu_c\cdot q_c(\mathbf{x})\right)d\mathbf{x}$$
$$= \sum_{j=1,j\neq i}^{N_{k+1|k}} r_j\cdot s_j + \mu_c\cdot s_c = \mu_{k+1} - r_i\cdot s_i$$

$$\Sigma_{k+1}^{\bar{i}} = \sum_{j=1,j\neq i}^{N_{k+1|k}}\left(r_j\cdot v_j - r_j^2\cdot s_j s_j T\right)$$
$$+\mu_c v_c - \left(\mu_c^2 - G^{(2)}(1)\right)s_c s_c^T$$
$$= \Sigma_{k+1} - \left(r_i\cdot v_i - r_i^2\cdot s_i s_i^T\right)$$

Now consider χ_A to be the the IIDC RFS with parameters $\{q_c(\mathbf{x}), \pi^c(n)\}$ and χ_B to be the multi-Bernoulli RFS with parameter set $\{r_j, q_j(\mathbf{x})\}_{j=1}^{N_{k+1|k}}$. In this case it can be shown that the multitarget density $\tilde{f}^A(W)$ corresponds to a IIDC RFS with parameters $\{q_c(\mathbf{x}), \frac{(n+1)!\cdot\pi^c(n+1)}{\mu_c}\}$ with probability 1. This is because, when $\mathbf{x}\notin W$ (event with probability 1), we have

$$\begin{aligned} \tilde{f}^A(W) &= \frac{f^A(\{\mathbf{x}\}\cup W)}{D^A(\mathbf{x})} \\ &= \frac{(|W|+1)\cdot\pi^c(|W|+1)\cdot q_c^{\{\mathbf{x}\}\cup W}}{\mu_c\cdot q_c(\mathbf{x})} \\ &= \frac{(|W|+1)\cdot\pi^c(|W|+1)\cdot q_c^W}{\mu_c} \\ &= \frac{(|W|+1)\cdot\pi^c(|W|+1)}{\mu_c}\cdot q_c^W \end{aligned}$$

The mean and second order moment of the cardinality distribution $\frac{(n+1)!\cdot\pi^c(n+1)}{\mu_c}$ in terms of the probability generating function of $\pi^c(n)$ are $\frac{G^{(2)}(1)}{\mu_c}$ and $\frac{G^{(3)}(1)}{\mu_c}$ respectively. Thus from equations (55) and (56) we have

$$\mu_{k+1}^{\bar{c}} = \sum_{j=1}^{N_{k+1|k}} r_j\cdot s_j + \frac{G^{(2)}(1)}{\mu_c}\cdot s_c,$$
$$\Sigma_{k+1}^{\bar{c}} = \sum_{j=1}^{N_{k+1|k}}\left(r_j\cdot v_j - r_j^2\cdot s_j s_j^T\right)$$

$$+\frac{G^{(2)}(1)}{\mu_c}v_c-\left(\frac{G^{(2)}(1)^2}{\mu_c^2}-\frac{G^{(3)}(1)}{\mu_c}\right)s_c s_c^T$$

Appendix D
Cardinality update for IIDC component

We find the cardinality distribution of the posterior IID cluster component. This can be defined as

$$\begin{aligned}
\pi^c_{k+1|k+1}(m_1) &= \int_{|W|^c=m_1} f_{k+1|k+1}(W)\delta W \\
&= \frac{\int_{|W|^c=m_1} h_{\mathbf{z}_{k+1}}(W) f_{k+1|k}(W)\delta W}{\int h_{\mathbf{z}_{k+1}}(W) f_{k+1|k}(W)\delta W} \\
&= \pi^c_{k+1|k}(m_1)\frac{\int h_{\mathbf{z}_{k+1}}(W) f^{c,m_1}_{k+1|k}(W)\delta W}{\int h_{\mathbf{z}_{k+1}}(W) f_{k+1|k}(W)\delta W}
\end{aligned}$$

where,

$$f^{c,m_1}_{k+1|k}(W)=\frac{1}{\pi^c_{k+1|k}(m_1)}\delta_{|W|^c}(m_1)f_{k+1|k}(W)$$

The multi-target density $f^{c,m_1}_{k+1|k}(W)$ corresponds to the union of independent multi-Bernoulli RFS and the random finite set obtained by constraining the cardinality ($|W|^c=m_1$) of the IID cluster RFS. Applying the approximations as before we get,

$$\pi^c_{k+1|k+1}(m_1)\approx\pi^c_{k+1|k}(m_1)\cdot\frac{\mathcal{N}_{\Sigma_r+\Sigma^{c,m_1}_{k+1}}(\mathbf{z}_{k+1}-\mu^{c,m_1}_{k+1})}{\mathcal{N}_{\Sigma_r+\Sigma_{k+1}}(\mathbf{z}_{k+1}-\mu_{k+1})}$$

where μ_{k+1} and Σ_{k+1} are as given in equations (55) and (56) and,

$$\begin{aligned}
\mu^{c,m_1}_{k+1} &= \sum_{i=1}^{N_{k+1|k}} r_i\cdot s_i+m_1\cdot s_c \\
\Sigma^{c,m_1}_{k+1} &= \sum_{i=1}^{N_{k+1|k}}\left(r_i\cdot v_i-r_i^2\cdot s_i s_i^T\right)+m_1(v_c-s_c s_c^T)
\end{aligned}$$

Note that in the above update equation there is no assumption made about the cardinality of the multi-Bernoulli component.

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