Multi-sensor PHD and CPHD filters

Santosh Nannuru, Mark Coates and Michael Rabbat

September 1, 2014

Abstract

We study various multi-sensor PHD and CPHD filters and their implementations.

1 Problem Statement

The state of the system is the collection of individual target states $\mathbf{x}_{k,i} \in R^{n_{\mathbf{x}}}$ and is denoted by the random finite set $X_k = \{\mathbf{x}_{k,1}, \ldots \mathbf{x}_{k,n_k}\}$ where $n_k \geq 0$ is number of targets present at time k. We assume that the individual target dynamics are specified according to the Markovian model of the form $\mathbf{x}_{k+1,i} = f_{k+1|k}(\mathbf{x}_{k,i}, \mathbf{u}_k)$ where $\mathbf{u}_k$ is the noise.

Information about the state of the system is available from sensors $1, \ldots, s$. These sensors make independent measurements and we denote $Z_{k,j}$ to be the measurement set of the j-th sensor at timestep k. Let $m_{k,j} = |Z_{k,j}|$. Let $Z_{[k],j}$ be the sequence of measurement sets at the j-th sensor, i.e. $Z_{1,j}, Z_{2,j}, \ldots, Z_{k,j}$.

Denote by $h_{k,j}(\mathbf{z}|\mathbf{x})$ the likelihood function at timestep k for sensor j for an individual measurement $\mathbf{z}$ and target state $\mathbf{x}$. Let the probability of detection be $p_{d,j}(\mathbf{x})$. Denote by $c_{k,j}(\mathbf{z})$ the clutter spatial distribution of the j-th sensor, and let $C_{k,j}(\mathbf{z})$ be the probability generating function (p.g.f.) of the cardinality distribution of the clutter process.

2 Product CPHD Filter

For deriving the approximate product multisensor CPHD filter we make the following modeling assumptions:

- $f_{k|k-1}(X)$ and $f_{k|k}(X|Z_{k,j})$ are the distributions of i.i.d. cluster processes.
- The sensor clutter processes are i.i.d. cluster processes.
- Each target generates at most one measurement per sensor.
- Each measurement is either associated with one target or is generated by clutter.

Abbreviate the multisensor PHD and multisensor cardinality distributions as:

$$D_{k|k}(\mathbf{x}) = D_{k|k}(\mathbf{x}|Z_{[k],1}, Z_{[k],2}, \dots Z_{[k],s}) \quad (1)$$
$$p_{k|k}(n) = p_{k|k}(n|Z_{[k],1}, Z_{[k],2}, \dots Z_{[k],s}) \quad (2)$$

We use the usual CPHD filter prediction equations to generate $D_{k+1|k}(\mathbf{x})$ and $p_{k+1|k}(n)$. Let

$$N_{k+1|k}(\mathbf{x}) = \int D_{k+1|k}(\mathbf{x})d\mathbf{x} \quad (3)$$
$$s_{k+1|k}(\mathbf{x}) = \frac{D_{k+1|k}(\mathbf{x})}{N_{k+1|k}(\mathbf{x})} \quad (4)$$
$$(5)$$

and let $G_{k+1|k}(y)$ be the p.g.f. of $p_{k+1|k}(n)$.

Mahler derives the following filter:

$$D_{k+1|k+1}(\mathbf{x}) = L_{Z_{k+1,1},\dots,Z_{k+1,s}}(\mathbf{x})\, s_{k+1|k}(\mathbf{x}) \quad (6)$$
$$p_{k+1|k+1}(n) = \frac{\widetilde{p}(n)\sigma^n}{\widetilde{G}(\sigma)} \quad (7)$$

2.1 Notation

Unfortunately, the terms in these update equations are fairly complicated.

Building up from the bottom:

- $\sigma_{m,i}(y_1,\dots,y_m)$ is the elementary symmetric function of degree i in m variables $y_1,\dots,y_m$ i.e. $\sigma_{m,i}(y_1,\dots,y_m) = \sum_{1\le j_1<j_2\cdots<j_i\le m} y_{j_1}y_{j_2}\dots y_{j_i}$.
- $f^{(i)}(y)$ denotes the i-th derivative of $f(y)$.
- $G(y) = \sum_{n\ge 0} p(n)y^n$ is the probability generating function (p.g.f.) of $p(n)$.
- For any function $h(\mathbf{x})$, let $s_{k+1|k}[h] = \int h(\mathbf{x})s_{k+1|k}(\mathbf{x})d\mathbf{x}$.
- Then $\gamma_j = s_{k+1|k}[1 - p_{d,j}]$.
- We use the abbreviated notation m_j for $m_{k,j}$
- Denote the observation at j^{th} sensor at time k as $Z_{k,j} = \{\mathbf{z}_{j,1}, \mathbf{z}_{j,2} \dots \mathbf{z}_{j,m_j}\}$. The time index has been dropped for brevity.

Let

$$\sigma_i(Z_{k,j}) = \sigma_{m_j,i}\left(\frac{s_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z}_{j,1})]}{c_{k+1}(\mathbf{z}_{j,1})}, \frac{s_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z}_{j,2})]}{c_{k+1}(\mathbf{z}_{j,2})}, \dots, \frac{s_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z}_{j,m_j})]}{c_{k+1}(\mathbf{z}_{j,m_j})}\right)$$

Let

$$\alpha_j(\mathbf{z}) = \frac{\sum_{\ell=0}^{m_j-1} C_{k+1,j}^{(m_j-\ell-1)}(0) G_{k+1|k}^{(\ell+1)}(\gamma_j)\sigma_\ell(Z_{k+1,j} - \{\mathbf{z}\})}{\sum_{i=0}^{m_j} C_{k+1,j}^{(m_j-i)}(0) G_{k+1|k}^{(i)}(\gamma_j)\sigma_i(Z_{k+1,j})} \tag{8}$$

$$\alpha_{j,0} = \frac{\sum_{\ell=0}^{m_j} C_{k+1,j}^{(m_j-\ell)}(0) G_{k+1|k}^{(\ell+1)}(\gamma_j)\sigma_\ell(Z_{k+1,j})}{\sum_{i=0}^{m_j} C_{k+1,j}^{(m_j-i)}(0) G_{k+1|k}^{(i)}(\gamma_j)\sigma_i(Z_{k+1,j})} \tag{9}$$

$$L_j(\mathbf{x}) = (1 - p_{d,j}(\mathbf{x}))\alpha_{j,0} + p_{d,j}(\mathbf{x}) \sum_{\mathbf{z}\in Z_{k+1,j}} h_{k,j}(\mathbf{z}|\mathbf{x}) \frac{\alpha_j(\mathbf{z})}{c_{k+1,j}(\mathbf{z})} \tag{10}$$

$$N_{k+1,j} = s_{k+1|k}[L_j] \tag{11}$$

$$\ell_j(n) = \sum_{i=0}^{\min(n,m_j)} \frac{n!}{(n-i)!} C_{k+1,j}^{(m_j-i)}(0)\, \gamma_j^{n-i} \sigma_i(Z_{k+1,j}) \tag{12}$$

$$\tilde{p}(n) = \ell_1(n)\ell_2(n)\ldots\ell_s(n)\, p_{k+1|k}(n) \tag{13}$$

$$\tilde{G}(y) = \sum_{n\geq 0} \tilde{p}(n) y^n \tag{14}$$

$$\sigma = \frac{s_{k+1|k}[L_1 L_2 \ldots L_s]}{N_{k+1,1}N_{k+1,2}\ldots N_{k+1,s}} \tag{15}$$

Then the multi-sensor pseudo-likelihood function is given by

$$L_{Z_{k+1,1},\ldots,Z_{k+1,s}}(\mathbf{x}) = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \frac{L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})}{N_{k+1,1}N_{k+1,2}\ldots N_{k+1,s}} \tag{16}$$

The PHD update equation for the multi-sensor CPHD filter is

$$D_{k+1|k+1}(\mathbf{x}) = L_{Z_{k+1,1},\ldots,Z_{k+1,s}}(\mathbf{x})\, s_{k+1|k}(\mathbf{x}) \tag{17}$$

$$= \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \times \frac{s_{k+1|k}(\mathbf{x})L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})}{N_{k+1,1}N_{k+1,2}\ldots N_{k+1,s}} \tag{18}$$

where we can identify, (19)

$$s_{k+1|k+1}(\mathbf{x}) = \frac{s_{k+1|k}(\mathbf{x})L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})}{s_{k+1|k}[L_1 L_2 \ldots L_s]} \tag{20}$$

$$N_{k+1|k+1} = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \times \frac{s_{k+1|k}[L_1 L_2 \ldots L_s]}{N_{k+1,1}N_{k+1,2}\ldots N_{k+1,s}} = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)}\sigma \tag{21}$$

The cardinality update equation for the multi-sensor CPHD filter is

$$p_{k+1|k+1}(n) = \frac{\widetilde{p}(n)\sigma^n}{\widetilde{G}(\sigma)} \tag{22}$$

Note that $N_{k+1|k+1}$ is the mean of the cardinality distribution $p_{k+1|k+1}(n)$.

2.2 Computational complexity

The computational complexity of calculating the elementary symmetric function $\sigma_{m,i}(y_1,\ldots,y_m)$ for $i = 1,2.\ldots m$ is of the order $\mathcal{O}(m^2)$ using a recursive method and can be reduced further to $\mathcal{O}(mlog^2m)$. For each sensor, computing L_j requires calculation of (m_j+1) different elementary symmetric functions. Thus computing L_j has complexity $\mathcal{O}(m_j^2 log^2 m_j)$. Thus the overall multi-sensor PHD update step has the complexity of $\mathcal{O}(\sum_{j=1}^{s} m_j^2 log^2 m_j)$. The complexity of the single-sensor CPHD filter is $\mathcal{O}(m^2 log^2 m)$.

Using a Gaussian assumption for the likelihood functions $h_{k,j}(\mathbf{z}|\mathbf{x})$ we can express $L_j(\mathbf{x})$ as sum of weighted Gaussians and a constant for each sensor. Modeling the predicted density $s_{k+1|k}(\mathbf{x})$ as a mixture of Gaussians, we can approximate the posterior PHD function $D_{k+1|k+1}(\mathbf{x})$ as a mixture of Gaussian densities. Assuming finite number of targets, we can propagate the posterior cardinality distribution $p_{k+1|k+1}(n)$.

3 Product PHD Filter

For deriving the approximate product multisensor PHD filter we make the following modeling assumptions:

- $f_{k|k-1}(X)$ is Poisson.
- The $f_{k|k}(X|Z_{k,j})$ are the distributions of i.i.d. cluster processes.
- The sensor clutter processes are Poisson. Let $c_{k,j}(\mathbf{z})$ be the clutter spatial distribution and $\lambda_{k,j}$ be the clutter rate of the j-th sensor.
- Each target generates at most one measurement per sensor.
- Each measurement is either associated with one target or is generated by clutter.

Abbreviate the posterior multisensor PHD distribution at time k as:

$$D_{k|k}(\mathbf{x}) = D_{k|k}(\mathbf{x}|Z_{[k],1}, Z_{[k],2}, \ldots Z_{[k],s}) \tag{23}$$

The predicted PHD at time $k+1$ is denoted by $D_{k+1|k}(\mathbf{x})$ and

$$N_{k+1|k}(\mathbf{x}) = \int D_{k+1|k}(\mathbf{x})d\mathbf{x} \tag{24}$$

$$s_{k+1|k}(\mathbf{x}) = \frac{D_{k+1|k}(\mathbf{x})}{N_{k+1|k}(\mathbf{x})} \tag{25}$$

Let

$$\sigma_i(Z_{k,j}) = \sigma_{m_j,i}\left(\frac{s_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z}_{j,1})]}{c_{k+1}(\mathbf{z}_{j,1})}, \frac{s_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z}_{j,2})]}{c_{k+1}(\mathbf{z}_{j,2})}, \ldots, \frac{s_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z}_{j,m_j})]}{c_{k+1}(\mathbf{z}_{j,m_j})}\right)$$

$$\gamma_j = s_{k+1|k}[1 - p_{d,j}] \tag{26}$$

$$L_j(\mathbf{x}) = (1 - p_{d,j}(\mathbf{x})) + p_{d,j}(\mathbf{x}) \sum_{\mathbf{z} \in Z_{k+1,j}} \frac{h_{k,j}(\mathbf{z}|\mathbf{x})}{\lambda_{k+1,J}c_{k+1}(z) + D_{k+1|k}[p_{d,j}h_{k,j}(\mathbf{z})]} \tag{27}$$

$$v_{k+1,j} = s_{k+1|k}[L_j] \tag{28}$$

$$\ell_j(n) = \sum_{i=0}^{\min(n,m_j)} \lambda_{k+1,j}^{m_j-i} \frac{n!}{(n-i)!} \gamma_j^{n-i} \sigma_i(Z_{k+1,j}) \tag{29}$$

$$\sigma = \frac{s_{k+1|k}[L_1 L_2 \ldots L_s]}{v_{k+1,1}v_{k+1,2}\cdots v_{k+1,s}} \tag{30}$$

$$\phi = \frac{\sum_{n\geq 0} \ell_1(n+1)\ell_2(n+1)\ldots\ell_s(n+1)\frac{(N_{k+1|k}\sigma)^n}{n!}}{\sum_{i\geq 0} \ell_1(i)\ell_2(i)\ldots\ell_s(i)\frac{(N_{k+1|k}\sigma)^i}{i!}} \tag{31}$$

Then the multi-sensor pseudo-likelihood function is given by

$$L_{Z_{k+1,1},\ldots,Z_{k+1,s}}(\mathbf{x}) = \phi \frac{L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})}{v_{k+1,1}v_{k+1,2}\cdots v_{k+1,s}} \tag{32}$$

The PHD update equation for the multi-sensor PHD filter is

$$D_{k+1|k+1}(\mathbf{x}) = L_{Z_{k+1,1},\ldots,Z_{k+1,s}}(\mathbf{x})\, D_{k+1|k}(\mathbf{x}) \tag{33}$$

$$= \phi \times \frac{D_{k+1|k}(\mathbf{x})L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})}{v_{k+1,1}v_{k+1,2}\cdots v_{k+1,s}} \tag{34}$$

where we can identify, (35)

$$s_{k+1|k+1}(\mathbf{x}) = \frac{s_{k+1|k}(\mathbf{x})L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})}{s_{k+1|k}[L_1 L_2 \ldots L_s]} \tag{36}$$

$$N_{k+1|k+1} = \phi \times N_{k+1|k} \times \frac{s_{k+1|k}[L_1 L_2 \ldots L_s]}{v_{k+1,1}v_{k+1,2}\cdots v_{k+1,s}} = N_{k+1|k}\,\phi\,\sigma \tag{37}$$

3.1 Analysis for two targets case

We perform simplification of the PHD product filter for the case when there are two targets within the monitoring region. Also assume there is no clutter ($\lambda = 0$) and probability of detection is very high ($p_d \approx 1$). Then from equation (29) we have

$$\ell_j(n) = \begin{cases} 0, & \text{if } n < m_j \\ m_j!\sigma_{m_j}(Z_{k+1,j}), & \text{if } n = m_j \\ \dfrac{n!}{(n-m_j)!}(1-p_d)^{n-m_j}\sigma_{m_j}(Z_{k+1,j}) \approx 0 & \text{if } n > m_j \end{cases} \tag{38}$$

Since $p_d \approx 1$, we have $m_j \approx N_{k+1|k+1}$. From the above simplification, we can approximately calculate

$$\phi \approx \frac{\ell_1(m_j)\ell_2(m_j)\ldots\ell_s(m_j)\dfrac{(N_{k+1|k}\sigma)^{(m_j-1)}}{(m_j-1)!}}{\ell_1(m_j)\ell_2(m_j)\ldots\ell_s(m_j)\dfrac{(N_{k+1|k}\sigma)^{m_j}}{m_j!}} \tag{39}$$

$$= \frac{m_j}{N_{k+1|k}}\frac{1}{\sigma} \approx \frac{N_{k+1|k+1}}{N_{k+1|k}}\frac{1}{\sigma} \tag{40}$$

From equation (37) we know that the above relation should hold exactly. Hence we should have $\ell_j(n)$ to be approximately zero except at $n = m_j \approx N_{k+1|k+1}$. This in turn depends on the value of $\sigma_{m_j}(Z_{k+1,j})$. The numerical evaluation of $\sigma_{m_j}(Z_{k+1,j})$ depends on the predicted PHD $s_{k+1|k}(\mathbf{x})$. Also the numerical evaluation of σ depends on the predicted PHD $s_{k+1|k}(\mathbf{x})$.

In a Gaussian mixture implementation, for the case of two targets and $p_d \approx 1$, we have $s_{k+1|k}(\mathbf{x})$ as a mixture of two Gaussian components with weights $w^{(1)}_{k+1|k}$ and $w^{(2)}_{k+1|k}$ and satisfying $w^{(1)}_{k+1|k} + w^{(2)}_{k+1|k} = 2$. The number of observations $m_j \approx 2$ and we can see from the definition of elementary symmetric function that

$$\sigma_2(Z_{k+1,j}) \propto w^{(1)}_{k+1|k} w^{(2)}_{k+1|k} \tag{41}$$

Hence when the two weights are not evenly distributed but still sum to 2 (Ex. $w^{(1)}_{k+1|k} = 0.001$, $w^{(1)}_{k+1|k} = 1.999$), the numerical value of $\sigma_{m_j}(Z_{k+1,j})$ decreases. We also have

$$s_{k+1|k}(\mathbf{x})[L_1(\mathbf{x})L_2(\mathbf{x})\ldots L_s(\mathbf{x})] \approx \sum_{z_1 \in Z_{k+1,1}} \cdots \sum_{z_s \in Z_{k+1,s}} \frac{s_{k+1|k}[h_1(z_1)\ldots h_s(z_s)]}{s_{k+1|k}[h_1(z_1)]\ldots s_{k+1|k}[h_s(z_s)]} \tag{42}$$

From above we can show that

$$\sigma \propto \frac{a_1}{(w^{(1)}_{k+1|k})^{(s-1)}} + \frac{a_2}{(w^{(2)}_{k+1|k})^{(s-1)}} \quad \text{where } a_1,\, a_2 \text{ are constants.} \tag{43}$$

Hence the numerical value of σ increases when the two weights are not evenly distributed. This causes more than one term to be significant in the summations in equation (31) for ϕ. This causes errors in the calculation of ϕ and hence error in estimation of $N_{k+1|k+1}$.

3.1.1 Analysis by Ouyang and Ji

Authors Ouyang and Ji in [1] have identified a similar problem with the product PHD filter when implemented using a SMC filter. Discuss their findings in detail.

3.2 Gaussian mixture implementation

Let the predicted PHD at time $k+1$ be given by the following Gaussian mixture

$$D_{k+1|k}(\mathbf{x}) = \sum_{i=1}^{J_{k+1|k}} w_{k+1|k}^{(i)} \mathcal{N}(\mathbf{x}, m_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}) \tag{44}$$

$$N_{k+1|k} = \sum_{i=1}^{J_{k+1|k}} w_{k+1|k}^{(i)} \tag{45}$$

We assume probability of detection is constant throughout the observation region so $p_{d,j}(\mathbf{x}) = p_{d,j}$ and that the clutter rate is the same for all sensors at all times, $\lambda_{k+1,j} = \lambda$. The pseudo code for the update step is provided in Figures 1 and 2.

Predicted parameters : $J_{k+1|k}, w_{k+1|k}^{(r)}, m_{k+1|k}^{(r)}, P_{k+1|k}^{(r)}$ for $r = 1, 2, \ldots J_{k+1|k}$

Measurements : $Z_{k+1,1}, Z_{k+1,2}, \ldots Z_{k+1,s}$

for $r = 1$ to $J_{k+1|k}$ **do**

  $\zeta^{(r)} = H_{k+1} m_{k+1|k}^{(r)}$

  $\Lambda^{(r)} = R_{k+1} + H_{k+1} P_{k+1|k}^{(r)} H_{k+1}^T$

end for

We build the product $D_{k+1|k}(\mathbf{x}) L_1(\mathbf{x}) L_2(\mathbf{x}) \ldots L_s(\mathbf{x})$ sequentially

Let : $w_{[0]}^{(r)} = w_{k+1|k}^{(r)}$, $m_{[0]}^{(r)} = m_{k+1|k}^{(r)}$, $P_{[0]}^{(r)} = P_{k+1|k}^{(r)}$, for $r = 1, 2, \ldots J_{k+1|k}$, $J_{[0]} = J_{k+1|k}$

for $j = 1$ to s **do** ▷ process each sensor information

  $\gamma_j = 1 - p_{d,j}$

  $\ell_j(n) = \sum_{i=0}^{\min(n,m_j)} \lambda^{m_j - i} \frac{n!}{(n-i)!} \gamma_j^{n-i} \sigma_i(Z_{k+1,j})$ for $n = 0, 1, 2, \ldots N_{max}$

  $v_{k+1,j} = (1 - p_{d,j}) + \frac{1}{N_{k+1|k}} \sum_{m=1}^{m_j} \sum_{r=1}^{J_{k+1|k}} \frac{p_{d,j} w_{k+1|k}^{(r)} \mathcal{N}(\mathbf{z}_{j,m}; \zeta^{(r)}, \Lambda^{(r)})}{\lambda\, c_{k+1}(\mathbf{z}_{j,m}) + \sum_{i=1}^{J_{k+1|k}} p_{d,j} w_{k+1|k}^{(i)} \mathcal{N}(\mathbf{z}_{j,m}; \zeta^{(i)}, \Lambda^{(i)})}$

  for $i = 1$ to $J_{[j-1]}$ **do** ▷ components corresponding to no detection

    $m_{[j]}^{(i)} = m_{[j-1]}^{(i)}$; $P_{[j]}^{(i)} = P_{[j-1]}^{(i)}$; $w_{[j]}^{(i)} = (1 - p_{d,j}) w_{[j-1]}^{(i)}$

    $\eta_{[j]}^{(i)} = H_{k+1} m_{[j-1]}^{(i)}$; $S_{[j]}^{(i)} = R_{k+1} + H_{k+1} P_{[j-1]}^{(i)} H_{k+1}^T$

    $K_{[j]}^{(i)} = P_{[j-1]}^{(i)} H_{k+1}^T [S_{[j]}^{(i)}]^{-1}$

  end for

  for $m = 1$ to m_j **do** ▷ components corresponding to detections

    for $i = 1$ to $J_{[j-1]}$ **do**

      $m_{[j]}^{(mJ_{[j-1]}+i)} = m_{[j-1]}^{(i)} + K_{[j]}^{(i)}(\mathbf{z}_{j,m} - \eta_{[j]}^{(i)})$

      $P_{[j]}^{(mJ_{[j-1]}+i)} = [I - K_{[j]}^{(i)} H_{k+1}] P_{[j-1]}^{(i)}$

      $w_{[j]}^{(mJ_{[j-1]}+i)} = \frac{p_{d,j} w_{[j-1]}^{(i)} \mathcal{N}(\mathbf{z}_{j,m}; \eta_{[j]}^{(i)}, S_{[j]}^{(i)})}{\lambda\, c_{k+1}(\mathbf{z}_{j,m}) + \sum_{r=1}^{J_{k+1|k}} p_{d,j} w_{k+1|k}^{(r)} \mathcal{N}(\mathbf{z}_{j,m}; \zeta^{(r)}, \Lambda^{(r)})}$

    end for

  end for

  $J_{[j]} = (m_j + 1) J_{[j-1]}$

end for

Figure 1: Pseudocode for product GM PHD filter

$N_{k+1|k} = \sum_{r=1}^{J_{k+1|k}} w_{k+1|k}^{(r)}$

$$\sigma = \frac{1}{N_{k+1|k}} \frac{\sum_{i=1}^{J_{[s]}} w_{[s]}^{(i)}}{v_{k+1,1} v_{k+1,2} \dots v_{k+1,s}}$$

$$\phi = \frac{\sum_{n \geq 0}^{N_{max}} \ell_1(n+1)\ell_2(n+1)\dots\ell_s(n+1)\frac{(N_{k+1|k}\sigma)^n}{n!}}{\sum_{i \geq 0}^{N_{max}} \ell_1(i)\ell_2(i)\dots\ell_s(i)\frac{(N_{k+1|k}\sigma)^i}{i!}}$$

Output posterior GM parameters :

for $i = 1$ to $J_{[s]}$ **do**

  $m_{k+1|k+1}^{(i)} = m_{[s]}^{(i)}; \quad P_{k+1|k+1}^{(i)} = P_{[s]}^{(i)}$

  $w_{k+1|k+1}^{(i)} = w_{[s]}^{(i)} \times \dfrac{\phi}{v_{k+1,1} v_{k+1,2} \dots v_{k+1,s}}$ ▷ final scaling of weights

end for

Figure 2: Pseudocode for product GM PHD filter (continued)

4 The general multi-sensor PHD filter

In this section we discuss the general multi-sensor PHD filter equations [2, 3]. The general multi-sensor PHD filter equations for the case of two sensors were first derived by Mahler in [2] and were later generalized for more than two sensors by Delande et al. in [3].

The derivation of the general multi-sensor PHD filter is based on the following assumptions (and notations)

- The predicted distribution at time $k+1$, $f_{k+1|k}(X)$ is Poisson.
- The sensor observation processes are independent conditional on the multitarget state.
- The sensor clutter processes are Poisson. Let $c_{k,j}(\mathbf{z}) = c_j(\mathbf{z})$ be the clutter spatial distribution and $\lambda_{k,j} = \lambda_j$ be the clutter rate of the j-th sensor.
- Each target generates at most one measurement per sensor.
- Each measurement is either associated with one target or is generated by clutter.
- If a target is present at location $\mathbf{x}$, sensor j detects it with probability $p_d^j(\mathbf{x})$ and generates a measurement $\mathbf{z}$ with probability density (likelihood function) given by $h_{k,j}(\mathbf{z}|\mathbf{x})$. Denote $q_d^j(\mathbf{x}) = 1 - p_d^j(\mathbf{x})$.

- Let $Z^j = \{\mathbf{z}_1^j, \mathbf{z}_2^j, \dots, \mathbf{z}_{m_j}^j\}$ be the observation set generated at time $k+1$ by sensor j and $|Z^j| = m_j$. Denote $Z^{(s)} = \bigcup_{j=1}^{s} Z^j$.

Let $\mathcal{S}$ be the collection of all s-ary partitons of $Z^{(s)}$. A partition P is said to be an s-ary partition of $Z^{(s)}$ if it consists of elements of the form $W = \{\mathbf{z}^{t_1}, \mathbf{z}^{t_2}, \dots, \mathbf{z}^{t_M}\}$ where, $1 \le t_1 < t_2 < \dots < t_M \le s$, $\mathbf{z}^{t_i} \in Z^{t_i}$, $M \in [1\ s]$ and if each element of $Z^{(s)}$ appears exactly once. We call W defined above as observation sequences. For example consider $s = 2$ and $Z^1 = \{\mathbf{z}_1^1, \mathbf{z}_2^1\}$, $Z^2 = \{\mathbf{z}_1^2, \mathbf{z}_2^2\}$. Then $Z^{(s)} = \{\mathbf{z}_1^1, \mathbf{z}_2^1, \mathbf{z}_1^2, \mathbf{z}_2^2\}$. The collection $\mathcal{S}$ has following 7 possible partitions

$$P_1 = \{\{\mathbf{z}_1^1\}, \{\mathbf{z}_2^1\}, \{\mathbf{z}_1^2\}, \{\mathbf{z}_2^2\}\}, \qquad P_2 = \{\{\mathbf{z}_1^1, \mathbf{z}_1^2\}, \{\mathbf{z}_2^1\}, \{\mathbf{z}_2^2\}\}, \quad P_3 = \{\{\mathbf{z}_1^1, \mathbf{z}_2^2\}, \{\mathbf{z}_2^1\}, \{\mathbf{z}_1^2\}\}, \tag{46}$$

$$P_4 = \{\{\mathbf{z}_2^1, \mathbf{z}_1^2\}, \{\mathbf{z}_1^1\}, \{\mathbf{z}_2^2\}\}, \qquad P_5 = \{\{\mathbf{z}_2^1, \mathbf{z}_2^2\}, \{\mathbf{z}_1^1\}, \{\mathbf{z}_1^2\}\}, \quad P_6 = \{\{\mathbf{z}_1^1, \mathbf{z}_1^2\}, \{\mathbf{z}_2^1, \mathbf{z}_2^2\}\}, \tag{47}$$

$$P_7 = \{\{\mathbf{z}_1^1, \mathbf{z}_2^2\}, \{\mathbf{z}_2^1, \mathbf{z}_1^2\}\}, \tag{48}$$

$$\mathcal{S} = \bigcup_{i=1}^{7} P_i \tag{49}$$

Let $D_{k+1|k}(\mathbf{x})$ and $D_{k+1|k+1}(\mathbf{x})$ be the predicted and posterior PHD functions.

$$\text{For } W = \{\mathbf{z}^{t_1}, \mathbf{z}^{t_2}, \dots, \mathbf{z}^{t_M}\}, \ \ 1 \le t_1 < t_2 < \dots < t_M \le s, \tag{50}$$

$$\mathbf{z}^{t_i} \in Z^{t_i} \text{ and } |W| = M \in [1\ s], \quad \text{define} \tag{51}$$

$$d_W = \begin{cases} D_{k+1|k}\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j \notin \{t_1,\dots,t_M\}} q_d^j\right)\right] & \text{if } M > 1 \\ \lambda_{t_1} c_{t_1}(\mathbf{z}^{t_1}) + D_{k+1|k}\left[p_d^{t_1}\, h_{t_1}(\mathbf{z}^{t_1})\left(\prod_{j \ne t_1} q_d^j\right)\right] & \text{if } M = 1 \end{cases} \tag{52}$$

$$\rho_W(\mathbf{x}) = \begin{cases} \dfrac{\left(\prod_{i=1}^{M} p_d^{t_i}(\mathbf{x})\, h_{t_i}(\mathbf{z}^{t_i}|\mathbf{x})\right)\left(\prod_{j \notin \{t_1,\dots,t_M\}} q_d^j(\mathbf{x})\right)}{D_{k+1|k}\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j \notin \{t_1,\dots,t_M\}} q_d^j\right)\right]} & \text{if } M > 1 \\ \dfrac{p_d^{t_1}(\mathbf{x})\, h_{t_1}(\mathbf{z}^{t_1}|\mathbf{x})\left(\prod_{j \ne t_1} q_d^j(\mathbf{x})\right)}{\lambda_{t_1} c_{t_1}(\mathbf{z}^{t_1}) + D_{k+1|k}\left[p_d^{t_1}\, h_{t_1}(\mathbf{z}^{t_1})\left(\prod_{j \ne t_1} q_d^j\right)\right]} & \text{if } M = 1 \end{cases} \tag{53}$$

Then the general multi-sensor PHD update equation is given by [2, 3]

$$\boxed{\frac{D_{k+1|k+1}(\mathbf{x})}{D_{k+1|k}(\mathbf{x})} = \prod_{j=1}^{s} q_d^j(\mathbf{x}) + \sum_{P \in \mathcal{S}} \frac{\prod_{W \in P} d_W}{\sum_{P \in \mathcal{S}} \prod_{W \in P} d_W}\left(\sum_{W \in P} \rho_W(\mathbf{x})\right)} \tag{54}$$

5 The general multi-sensor CPHD filter

In this section we derive the general multi-sensor CPHD filter. For deriving the general multi-sensor CPHD filter we make the following modeling assumptions:

- The predicted distribution at time $k+1$, $f_{k+1|k}(X)$ is IIDC.
- The predicted spatial distribution is denoted by $s_{k+1|k}(\mathbf{x}) = s(\mathbf{x})$ and PGF of the predicted cardinality distribution is denoted by $G_{k+1|k}(t) \equiv G(t)$.
- For an IIDC process with PGFL $G[h]$ and PGF $G(t)$, we know that $G[h] = G(s[h])$.
- The sensor observation processes are independent conditional on the multitarget state X.
- The sensor clutter processes are IIDC. Let $c_{k+1,j}(\mathbf{z}) = c_j(\mathbf{z})$ be the clutter spatial distribution and $C_{k+1,j}(t) = C_j(t)$ be the PGF of the clutter cardinality distribution of the j^{th} sensor, $j = 1, 2, \ldots, s$.
- Each target generates at most one measurement per sensor
- Each measurement is either associated with one target or is generated by clutter.
- If a target is present at location $\mathbf{x}$, sensor j detects it with probability $p_d^j(\mathbf{x})$ and generates a measurement $\mathbf{z}$ with probability density (likelihood function) given by $h_j(\mathbf{z}|\mathbf{x})$.
- Let Z^j be the observation set generated at time $k+1$ by sensor j. Also let $|Z^j| = m_j$.
- Denote $q_d^j(\mathbf{x}) = 1 - p_d^j(\mathbf{x})$.

We define the multivariate functional $F[g_1, g_2, \ldots, g_s, h]$ as follows

$$F[g_1, g_2, \ldots, g_s, h] = \int h^X \, G_{k+1,1}[g_1|X] \, G_{k+1,2}[g_2|X] \ldots G_{k+1,s}[g_s|X] \, f_{k+1|k}(X) \, \delta X \tag{55}$$

$$G_{k+1,j}[g_j|X] = \int g_j^Z \, f_{k+1,j}(Z|X) \delta Z \tag{56}$$

where $f_{k+1,j}(Z|X)$ is the multitarget likelihood function for the j^{th} sensor.

From the assumptions on the observation model and clutter process, we can show that [4]

$$G_{k+1,j}[g_j|X] = \int g_j^Z \, f_{k+1,j}(Z|X) \delta Z, \; j = 1, 2, \ldots, s \tag{57}$$

$$= C_j(c_j[g_j]) \, (1 - p_d^j + p_d^j \, p_{g_j})^X \tag{58}$$

$$= C_j(c_j[g_j]) \, \phi_{g_j}^X \tag{59}$$

$$\text{where, } \phi_{g_j}(\mathbf{x}) = 1 - p_d^j(\mathbf{x}) + p_d^j(\mathbf{x}) \, p_{g_j}(\mathbf{x}), \quad j = 1, 2, \ldots, s \tag{60}$$

$$p_{g_j}(\mathbf{x}) = \int g_j(\mathbf{z}) h_j(\mathbf{z}|\mathbf{x}) d\mathbf{z}, \quad j = 1, 2, \ldots, s. \tag{61}$$

Using the above result in equation (55) we have

$$F[g_1, g_2, \ldots, g_s, h] = \int h^X \left(\prod_{j=1}^{s} C_j(c_j[g_j])\right)\left(\prod_{j=1}^{s} \phi_{g_j}^X\right) f_{k+1|k}(X)\,\delta X \tag{62}$$

$$= \left(\prod_{j=1}^{s} C_j(c_j[g_j])\right) \int \left(h \prod_{j=1}^{s} \phi_{g_j}\right)^X f_{k+1|k}(X)\,\delta X \tag{63}$$

$$= \left(\prod_{j=1}^{s} C_j(c_j[g_j])\right) G(s[h \prod_{j=1}^{s} \phi_{g_j}]) \tag{64}$$

The last step results from the assumption that the predicted multitarget distribution $f_{k+1|k}(X)$ is IIDC.

Let $f_{k+1|k+1}(X)$ be the posterior multitarget density at time $k+1$. Let $G_{k+1|k+1}[h]$ be the PGFL of the multitarget density $f_{k+1|k+1}(X)$ and $D_{k+1|k+1}(\mathbf{x})$ be the posterior PHD. Then it can be shown that [2, 3],

$$G_{k+1|k+1}[h] = \frac{\dfrac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, h]}{\dfrac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, 1]} \tag{65}$$

$$D_{k+1|k+1}(\mathbf{x}) = \frac{\delta G_{k+1|k+1}}{\delta \mathbf{x}}[1] = \frac{\dfrac{\delta F}{\delta \mathbf{x} \delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, 1]}{\dfrac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, 1]} \tag{66}$$

Note that the differentiation δZ^j is with respect to the function variable g_j. The differentiation $\delta \mathbf{x}$ is with respect to the function variable h.

We now define the following notation

$$C_j^{(i)}(t) = C_{k+1,j}^{(i)}(t) = \frac{d^i C_{k+1,j}}{dt^i}(t), \quad j = 1, 2, \ldots, s \tag{67}$$

$$G^{(i)}(t) = G_{k+1|k}^{(i)}(t) = \frac{d^i G_{k+1|k}}{dt^i}(t) \tag{68}$$

$$Z^{(s)} = \bigcup_{j=1}^{s} Z^j \tag{69}$$

$$\mathcal{S} = \text{collection of all } s-\text{ary partitons of all possible subsets of } Z^{(s)} \tag{70}$$

Using mathematical induction on the number of sensors, we can show that

$$\boxed{\frac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^s}[g_1, g_2, \ldots, g_s, h] = \Gamma \sum_{P \in \mathcal{S}} \psi_P[g_1, g_2, \ldots, g_s, h] \prod_{W \in P} d_W[g_1, g_2, \ldots, g_s, h]} \tag{71}$$

$$\text{where, } \Gamma = \prod_{j=1}^{s} \left(\prod_{\mathbf{z} \in Z^j} c_j(\mathbf{z}) \right) \tag{72}$$

$$\psi_P[g_1, g_2, \ldots, g_s, h] = \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(c_j[g_j]) \right) G^{(|P|)}(s[h \prod_{j=1}^{s} \phi_{g_j}]) \tag{73}$$

$$|P| = \text{cardinality of the } s\text{-ary partition P} \tag{74}$$

$$|P|_j = \text{number of elements of P containing observation from sensor } j \tag{75}$$

$$\text{For } 1 \le t_1 < t_2 < \cdots < t_M \le s, \text{ let } W = \{\mathbf{z}^{t_1}, \mathbf{z}^{t_2}, \ldots, \mathbf{z}^{t_M}\} \tag{76}$$

$$\text{where, } \mathbf{z}^{t_i} \in Z^{t_i} \text{ and } |W| = M \in [1 \; s], \quad \text{then} \tag{77}$$

$$d_W[g_1, g_2, \ldots, g_s, h] = \frac{s\left[h \left(\prod_{i=1}^{M} p_d^{t_i} \, h_{t_i}(\mathbf{z}^{t_i}) \right) \left(\prod_{j \notin \{t_1, \ldots, t_M\}} \phi_{g_j} \right) \right]}{\prod_{i=1}^{M} c_{t_i}(\mathbf{z}^{t_i})} \tag{78}$$

The proof of the above result is provided in Appendix B. We now give an example listing all the partitions for $s = 2$. Let $Z^1 = \{\mathbf{z}_1^1, \mathbf{z}_2^1\}$ and $Z^2 = \{\mathbf{z}_1^2\}$. Hence the collection $\mathcal{S}$ has the following 12 possible partitions (generalized partitions)

$$\begin{aligned} \mathcal{S} = \{ \; \phi, \quad & \{\{\mathbf{z}_1^1\}\}, \quad \{\{\mathbf{z}_2^1\}\}, \quad \{\{\mathbf{z}_1^2\}\}, \quad \{\{\mathbf{z}_1^1\},\{\mathbf{z}_2^1\}\}, \quad \{\{\mathbf{z}_1^1\},\{\mathbf{z}_1^2\}\}, \quad \{\{\mathbf{z}_2^1\},\{\mathbf{z}_1^2\}\}, \\ & \{\{\mathbf{z}_1^1\},\{\mathbf{z}_2^1\},\{\mathbf{z}_1^2\}\}, \quad \{\{\mathbf{z}_1^1,\mathbf{z}_1^2\},\{\mathbf{z}_2^1\}\}, \quad \{\{\mathbf{z}_1^1,\mathbf{z}_1^2\}\}, \quad \{\{\mathbf{z}_1^1\},\{\mathbf{z}_2^1,\mathbf{z}_1^2\}\}, \quad \{\{\mathbf{z}_2^1,\mathbf{z}_1^2\}\} \; \}. \end{aligned} \tag{79}$$

Substituting $g_j \equiv 0;\; j = 1, 2, \ldots, s$ we get

$$\frac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, h] = \Gamma \sum_{P \in \mathcal{S}} \psi_P[0, 0, \ldots, 0, h] \prod_{W \in P} d_W[0, 0, \ldots, 0, h] \tag{80}$$

We can now differentiate the above expression and substitute $h \equiv 1$ to get

$$\frac{\delta F}{\delta \mathbf{x} \delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, 1] = \Gamma \sum_{P \in \mathcal{S}} \left(\psi_P^* \prod_{W \in P} d_W \right) \times \left(s(\mathbf{x}) \prod_{j=1}^{s} q_d^j(\mathbf{x}) \right) \tag{81}$$

$$+ \quad \Gamma \sum_{P \in \mathcal{S}} \left(\psi_P \prod_{W \in P} d_W \right) \times \left(s(\mathbf{x}) \sum_{W \in P} \rho_W(\mathbf{x}) \right) \tag{82}$$

$$\text{where,} \tag{83}$$

$$\psi_P^* = \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|+1)}(s[\prod_{j=1}^{s} q_d^j]) \tag{84}$$

$$\psi_P = \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|)}(s[\prod_{j=1}^{s} q_d^j]) \tag{85}$$

$$\text{For } 1 \le t_1 < t_2 < \cdots < t_M \le s, \text{ let } W = \{\mathbf{z}^{t_1}, \mathbf{z}^{t_2}, \ldots, \mathbf{z}^{t_M}\} \tag{86}$$

$$\text{where, } \mathbf{z}^{t_i} \in Z^{t_i} \text{ and } |W| = M \in [1\ s], \quad \text{then} \tag{87}$$

$$d_W = \frac{s\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j\notin\{t_1,\ldots,t_M\}} q_d^j\right)\right]}{\prod_{i=1}^{M} c_{t_i}(\mathbf{z}^{t_i})} \tag{88}$$

$$\rho_W(\mathbf{x}) = \frac{\left(\prod_{i=1}^{M} p_d^{t_i}(\mathbf{x})\, h_{t_i}(\mathbf{z}^{t_i}|\mathbf{x})\right)\left(\prod_{j\notin\{t_1,\ldots,t_M\}} q_d^j(\mathbf{x})\right)}{s\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j\notin\{t_1,\ldots,t_M\}} q_d^j\right)\right]} \tag{89}$$

Proof of above result is given in Appendix C. We also have

$$\frac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^s}[0, 0, \ldots, 0, 1] = \Gamma \sum_{P\in\mathcal{S}} \psi_P \prod_{W\in P} d_W \tag{90}$$

Using the above results and the definition of PHD from equation 66 we have the following update equation for PHD

$$\boxed{\frac{D_{k+1|k+1}(\mathbf{x})}{s_{k+1|k}(\mathbf{x})} = \frac{\sum_{P\in\mathcal{S}}\left(\psi_P^* \prod_{W\in P} d_W\right)}{\sum_{P\in\mathcal{S}}\left(\psi_P \prod_{W\in P} d_W\right)} \prod_{j=1}^{s} q_d^j(\mathbf{x}) + \sum_{P\in\mathcal{S}} \frac{\psi_P \prod_{W\in P} d_W}{\sum_{P\in\mathcal{S}}\left(\psi_P \prod_{W\in P} d_W\right)} \left(\sum_{W\in P} \rho_W(\mathbf{x})\right)} \tag{91}$$

The update equation for the posterior cardinality distribution $p_{k+1|k+1}(n)$ can be shown to be

$$\boxed{\frac{p_{k+1|k+1}(n)}{p_{k+1|k}(n)} = \frac{\sum_{\substack{P\in\mathcal{S}\\|P|\le n}} \frac{n!}{(n-|P|)!}\left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right)(s[\prod_{j=1}^{s} q_d^j])^{n-|P|} \prod_{W\in P} d_W}{\sum_{P\in\mathcal{S}}\left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right) G^{(|P|)}(s[\prod_{j=1}^{s} q_d^j]) \prod_{W\in P} d_W}} \tag{92}$$

Derivation of above expression is given in Appendix D.

6 Implementation of multi-sensor PHD and CPHD filters

An implementation of the general multi-sensor PHD filter based on Gaussian mixture model is discussed in [5]. The authors implement the general s-sensor PHD filter by repeated application of the two-sensor PHD filter. An approach to reduce the number of partitions for computational tractability is also suggested. The number of partitions are restricted by retaining only a fraction of the observations from each sensor for each predicted target. This observation selection is done using the nearest neighbor principle.

In this section we present a different method to reduce the number of partitions. We focus on limiting the total number of observation sequences W rather than restricting the observations from each sensor. Restricting the number of partitions is important for computational tractability of the algorithms. The proposed approach is discussed next.

Consider the observation sequence $W = \{\mathbf{z}^{t_1}, \mathbf{z}^{t_2}, \dots, \mathbf{z}^{t_M}\}$ where, $1 \leq t_1 < t_2 < \dots < t_M \leq s$, $\mathbf{z}^{t_i} \in Z^{t_i}$, $M \in [1\ s]$. We now associate a weight β with this observation sequence (this weight can be intuitively thought of as the likelihood that this observation sequence was generated by a single target source) which is defined as follows

$$\beta(W) = D_{k+1|k}\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j\notin\{t_1,\dots,t_M\}} q_d^j\right)\right] \tag{93}$$

The predicted PHD is represented by the Gaussian mixture

$$D_{k+1|k}(\mathbf{x}) = \sum_{i=1}^{J_{k+1|k}} w_{k+1|k}^{(i)} \mathcal{N}(\mathbf{x}, m_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}) \tag{94}$$

Thus we have

$$\beta(W) = \sum_{i=1}^{J_{k+1|k}} w_{k+1|k}^{(i)} \mathcal{N}(\cdot; m_{k+1|k}^{(i)}, P_{k+1|k}^{(i)})\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j\notin\{t_1,\dots,t_M\}} q_d^j\right)\right] \tag{95}$$

$$= \sum_{i=1}^{J_{k+1|k}} \beta^{(i)}(W) \tag{96}$$

$$\beta^{(i)}(W) = w_{k+1|k}^{(i)} \mathcal{N}(\cdot; m_{k+1|k}^{(i)}, P_{k+1|k}^{(i)})\left[\left(\prod_{i=1}^{M} p_d^{t_i}\, h_{t_i}(\mathbf{z}^{t_i})\right)\left(\prod_{j\notin\{t_1,\dots,t_M\}} q_d^j\right)\right] \tag{97}$$

$\beta^{(i)}(W)$ is the weight associated with the observation sequence W for the i^{th} Gaussian component (target). We use $\beta^{(i)}(W)$ to rank observation sequences for each Gaussian component and retain only a fraction of them corresponding to highest weights. We can calculate $\beta^{(i)}(W)$ for all possible observation sequences W. The total number of such possible observation sequences is $\prod_{j=1}^{s}(1 + m_j)$. This number can be very large if the clutter rate is high. But a large fraction of these will have

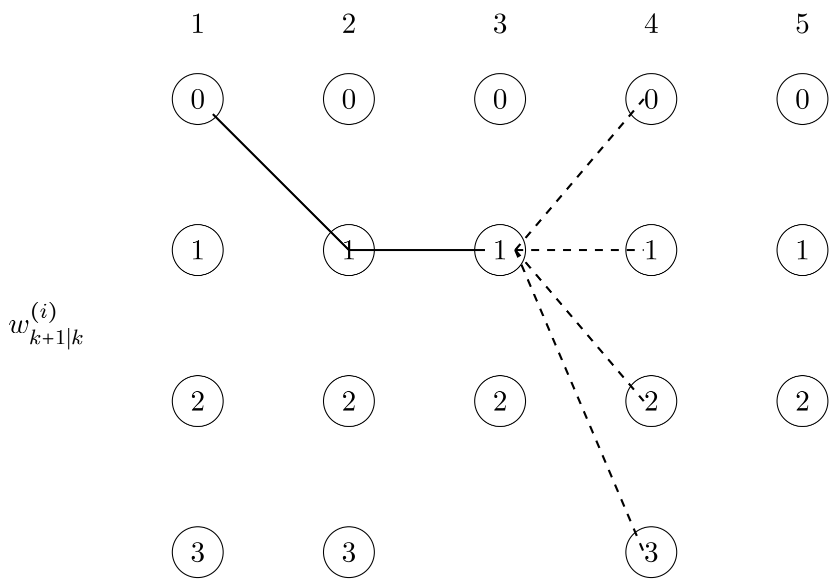

Figure 3: Trellis diagram

very low values. Hence we propose the following method for calculating $\beta^{(i)}(W)$ and retaining W with highest weights.

When the sensor observation noise is Gaussian, value of $\beta^{(i)}(W)$ can be sequentially calculated. Thus we can incrementally incorporate information from newer sensors. The computation of $\beta^{(i)}(W)$ is done by processing one sensor at a time. After processing each sensor, we retain only a fraction ($P_{\max}$) of the possible observation sequences based on the highest $\beta^{(i)}(W)$ values. Thus in the end we only have a small number of observation sequences left. The above method will give us a collection of observation sequences for each existing Gaussian component. All these individual collections are combined to give a single set of observation sequences. Valid 's-ary' partitions are then formed from these observation sequences.

The above algorithm can be visualized in the form of a trellis diagram. The nodes of the trellis are the sensor observations (or the no detection event). A pictorial representation is given in Figure 3. Each column of trellis corresponds to observations from one of the sensors. The no detection event is denote by node 0. Sensor number is indicated at top of each column. Each path through the trellis corresponds to a different observation sequence. The sequential sensor processing can be demonstrated as follows. An observation sequence (path) retained after processing observations from sensor 3 is shown as a solid line passing through nodes 0, 1 and 1 corresponding to sensors 1,2 and 3 respectively. When processing sensor 4 information, this path is extended for each node of sensor 4 as represented by the dashed line. The weights of these new paths are calculated using the expression for $\beta^{(i)}(W)$ but limited to only first 4 sensors. This is done for each existing path for each of the Gaussian components. Only $P_{\max}$ paths are retained for each component corresponding to $P_{\max}$ highest weights.

Note that we cannot optimally implement a Viterbi type algorithm for this trellis because the incremental weight update calculation of $\beta^{(i)}(W)$ is path (W) dependent. A pseudocode of the proposed algorithm to identify the best observation sequences is given in Figure 4. Pseudocode of the general multisensor PHD filter based on these observation sequences is given in Figure 5. The pseudocode for the general multisensor CPHD filter update step is given in Figure 6. Note that for the CPHD filter we use the generalized partition definition given in Section 5 (for example see equation 79). Given the set of partitions we can easily expand it to obtain the set of generalized partitions.

The problem of finding all possible partitions from a given collection of paths can be mapped to the exact cover problem in computer science [6]. An efficient algorithm called Dancing Links has been suggested by Donald Knuth [7] for solving this problem. We use an open source implementation of this algorithm in C programming language which is available here [8]. The implementation uses a doubly linked list to represent the collection of paths and the partitions are constructed by invoking a recursive function. The exact cover problem is classified as a NP-complete problem in literature [6].

Intuitively a partition is a grouping of observations such that elements of the partition are generated by a single source (target) or clutter. Each singleton element of the partition corresponds to an observation either being generated by a target or by the clutter process of the corresponding sensor. Each non-singleton element of the partition corresponds to a target which has been detected by at least two sensors. Non-singleton elements cannot correspond to clutter since at least two sensors have detected them.

We illustrate the concept of partitions using the following example for the case when there are $s = 2$ sensors. Let the sensor observations be $Z^1 = \{\mathbf{z}_1^1, \mathbf{z}_2^1, \mathbf{z}_3^1, \mathbf{z}_4^1\}$ and $Z^2 = \{\mathbf{z}_1^2, \mathbf{z}_2^2\}$. After application of Algorithm in Figure 4 to identify the best observation sequences lets assume that the following sequences were considered to be significant $\{\mathbf{z}_1^1\}; \{\mathbf{z}_3^1\}; \{\mathbf{z}_1^2\}; \{\mathbf{z}_2^2\}; \{\mathbf{z}_1^1, \mathbf{z}_1^2\}; \{\mathbf{z}_3^1, \mathbf{z}_2^2\}$. For this set of observation sequences we have 3 possible partitions which are listed as following

$$P_1 = \{\{\mathbf{z}_1^1\}, \{\mathbf{z}_3^1\}, \{\mathbf{z}_1^2\}, \{\mathbf{z}_2^2\}\}; \quad P_2 = \{\{\mathbf{z}_3^1\}, \{\mathbf{z}_2^2\}, \{\mathbf{z}_1^1, \mathbf{z}_1^2\}\}; \tag{98}$$

$$P_3 = \{\{\mathbf{z}_1^1\}, \{\mathbf{z}_1^2\}, \{\mathbf{z}_3^1, \mathbf{z}_2^2\}\}; \tag{99}$$

Observations $\mathbf{z}_2^1$ and $\mathbf{z}_4^1$ which do not appear in the partitions are excluded from further processing.

Since we do not consider all the possible observation sequences while constructing partitions it might be the case that the selected sequences do not permit any partition. For example consider the case when there are $s = 2$ sensors and observations Z^1 and Z^2 are as given above. Assume that the following sequences were considered to be significant $\{\mathbf{z}_1^1\}; \{\mathbf{z}_1^1, \mathbf{z}_1^2\}; \{\mathbf{z}_1^1, \mathbf{z}_2^2\}$. Note that we cannot form any partition from the above set of sequences. To resolve this issue we introduce singleton observation sequences corresponding to each included observation in the identified sequences. For example, in the above case we include the singleton sequences $\{\mathbf{z}_1^2\}$ and $\{\mathbf{z}_2^2\}$. We can now form the following 3 partitions

$$P_1 = \{\{\mathbf{z}_1^1\}, \{\mathbf{z}_1^2\}, \{\mathbf{z}_2^2\}\}; \quad P_2 = \{\{\mathbf{z}_2^2\}, \{\mathbf{z}_1^1, \mathbf{z}_1^2\}\}; \quad P_3 = \{\{\mathbf{z}_1^2\}, \{\mathbf{z}_1^1, \mathbf{z}_2^2\}\}; \tag{100}$$

Predicted parameters : $J_{k+1|k}, w^{(r)}_{k+1|k}, m^{(r)}_{k+1|k}, P^{(r)}_{k+1|k}$ for $r = 1, 2, \dots J_{k+1|k}$

Measurements : $Z^1_{k+1}, Z^2_{k+1}, \dots Z^s_{k+1}$

for $r = 1$ to $J_{k+1|k}$ **do**

  Let : $w^{(1)}_{[r,0]} = w^{(r)}_{k+1|k}$, $m^{(1)}_{[r,0]} = m^{(r)}_{k+1|k}$, $P^{(1)}_{[r,0]} = P^{(r)}_{k+1|k}$, $J_{[r,0]} = 1$, $\text{paths}^{(1)}_{[r,0]} = \phi$

  for $j = 1$ to s **do** ▷ process each sensor information

    for $i = 1$ to $J_{[r,j-1]}$ **do** ▷ process no detection event for each existing path

      $w^{(i)}_{[r,j]} = (1 - p_{d,j}) w^{(i)}_{[r,j-1]}$

      $m^{(i)}_{[r,j]} = m^{(i)}_{[r,j-1]}$, $P^{(i)}_{[r,j]} = P^{(i)}_{[r,j-1]}$

      $\text{paths}^{(i)}_{[r,j]} = \text{append}(\text{paths}^{(i)}_{[r,j-1]}, -1)$ ▷ -1 corresponds to no detection event

      $\eta^{(i)}_{[r,j]} = H_{k+1} m^{(i)}_{[r,j-1]}$; $S^{(i)}_{[r,j]} = R_{k+1} + H_{k+1} P^{(i)}_{[r,j-1]} H^T_{k+1}$ ▷ intermediate parameters

      $K^{(i)}_{[r,j]} = P^{(i)}_{[r,j-1]} H^T_{k+1} [S^{(i)}_{[r,j]}]^{-1}$

    end for

    for $m = 1$ to m_j **do** ▷ components corresponding to detections

      for $i = 1$ to $J_{[r,j-1]}$ **do** ▷ for each existing path

        $w^{(mJ_{[r,j-1]}+i)}_{[r,j]} = p_{d,j}\, w^{(i)}_{[r,j-1]} \mathcal{N}(\mathbf{z}^j_m; \eta^{(i)}_{[r,j]}, S^{(i)}_{[r,j]})$

        $m^{(mJ_{[r,j-1]}+i)}_{[r,j]} = m^{(i)}_{[r,j-1]} + K^{(i)}_{[r,j]}(\mathbf{z}^j_m - \eta^{(i)}_{[r,j]})$

        $P^{(mJ_{[r,j-1]}+i)}_{[r,j]} = [I - K^{(i)}_{[r,j]} H_{k+1}] P^{(i)}_{[r,j-1]}$

        $\text{paths}^{(mJ_{[r,j-1]}+i)}_{[r,j]} = \text{append}(\text{paths}^{(i)}_{[r,j-1]}, m)$

      end for

    end for

    $J_{[r,j]} = \min(P_{max}, (1 + m_j) J_{[r,j-1]})$ ▷ retain up to P_{max} paths with highest weights

    $\text{Idx} = \text{sort descending}(w_{[r,j]})$

    $w_{[r,j]} = w^{(\text{Idx}(1:J_{[r,j]}))}_{[r,j]}$

    $m_{[r,j]} = m^{(\text{Idx}(1:J_{[r,j]}))}_{[r,j]}$, $P_{[r,j]} = P^{(\text{Idx}(1:J_{[r,j]}))}_{[r,j]}$

    $\text{paths}_{[r,j]} = \text{paths}^{(\text{Idx}(1:J_{[r,j]}))}_{[r,j]}$

  end for

end for

Output : $\text{paths}_{[r,s]}$ for $r = 1, 2, \dots J_{k+1|k}$

Figure 4: Pseudocode for identifying best observation sequences

Selected paths : paths $= \bigcup_{r=1}^{J_{k+1|k}} \text{paths}_{[r,s]}$ ▷ Output of Algorithm in Figure 4

$\mathcal{S}$ = partitions(paths) ▷ Set of all partitions

PHD update : $D_{k+1|k+1}(\mathbf{x}) = D_{k+1|k}(\mathbf{x})\,\Omega_\phi(\mathbf{x}) + \sum_{W \in \text{paths}} D_{k+1|k}(\mathbf{x})\,\Omega_W(\mathbf{x})$

$\Omega_\phi(\mathbf{x}) = \prod_{j=1}^{s} q_d^j(\mathbf{x})$; $\Omega_W(\mathbf{x}) = \dfrac{\sum_{\mathcal{P} \in S_W} \prod_{W \in \mathcal{P}} d_W}{\sum_{\mathcal{P} \in S} \prod_{W \in \mathcal{P}} d_W}\, \rho_W(\mathbf{x})$

S_W = set of all partitions including the path W, $S_W \subseteq S$

d_W and $\rho_W(\mathbf{x})$ are as defined in equations (52) and (53)

Figure 5: Pseudocode for general multisensor PHD filter (update step)

Selected paths : paths $= \bigcup_{r=1}^{J_{k+1|k}} \text{paths}_{[r,s]}$ ▷ Output of Algorithm in Figure 4

$\mathcal{S}$ = generalized partitions(paths) ▷ Set of all generalized partitions

PHD update : $D_{k+1|k+1}(\mathbf{x}) = s_{k+1|k}(\mathbf{x})\,\Omega_\phi(\mathbf{x}) + \sum_{W \in \text{paths}} s_{k+1|k}(\mathbf{x})\,\Omega_W(\mathbf{x})$

$\Omega_\phi(\mathbf{x}) = \dfrac{\sum_{P \in \mathcal{S}} \left(\psi_P^* \prod_{W \in P} d_W \right)}{\sum_{P \in \mathcal{S}} \left(\psi_P \prod_{W \in P} d_W \right)} \prod_{j=1}^{s} q_d^j(\mathbf{x})$; $\Omega_W(\mathbf{x}) = \dfrac{\sum_{\mathcal{P} \in S_W} \left(\psi_P \prod_{W \in P} d_W \right)}{\sum_{P \in \mathcal{S}} \left(\psi_P \prod_{W \in P} d_W \right)}\, \rho_W(\mathbf{x})$

S_W = set of all partitions which include the path W, $S_W \subseteq S$

d_W and $\rho_W(\mathbf{x})$ are as defined in equations (88) and (89)

Cardinality update : $p_{k+1|k+1}(n) = p_{k+1|k}(n)\,\alpha(n)$

$\alpha(n) = \dfrac{\sum_{\substack{P \in \mathcal{S} \\ |P| \le n}} \left(\psi_P^n \prod_{W \in P} d_W \right)}{\sum_{P \in \mathcal{S}} \left(\psi_P \prod_{W \in P} d_W \right)}$; $\psi_P^n = \dfrac{n!}{(n-|P|)!} \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) (s[\prod_{j=1}^{s} q_d^j])^{n-|P|}$

Figure 6: Pseudocode for general multisensor CPHD filter (update step)

6.1 Implementation of CPHD filter

We propose the following algorithm to incrementally build partial covers. Let $\text{paths}_{[r]}$ denote the collection of paths for the r^{th} Gaussian component. Let $\mathcal{S}_{(r)}$ be the collection of partial covers obtained after processing the r^{th} component. Denote $\mathcal{S}_{(r)} = \{P^1_{(r)}, P^2_{(r)}, \dots\}$ and $\text{paths}_{[r]} = \{W^1_r, W^2_r, \dots\}$. Let $\beta^1_{(r)}, \beta^2_{(r)}, \dots$ be the weights associated with the partial covers $P^1_{(r)}, P^2_{(r)}, \dots$. We retain a maximum of $J_{\max}$ partial covers after processing each component. A pseudocode of the proposed algorithm is provided in Figure 7.

Input : $\text{paths}_{[r]},\ r = 1, 2, \dots, J_{k+1|k}$
$\mathcal{S}_{(0)} = \{\phi\},\ |\mathcal{S}_{(0)}| = 1,\ \beta^1_{(0)} = 1$
for $r = 1$ to $J_{k+1|k}$ **do** ▷ for each Gaussian component
  $j = 0$
  for $i = 1$ to $|\mathcal{S}_{(r-1)}|$ **do** ▷ for each existing partial cover
    for $l = 1$ to $|\text{paths}_{[r]}|$ **do** ▷ for each potential path
      if $P^i_{(r-1)} \cap W^l_r = \phi$ **then**
        $j = j + 1$
        $P^j_{(r)} = P^i_{(r-1)} \cup W^l_r$ ▷ increment the partial cover
        $\beta^j_{(r)} = \beta^i_{(r-1)} \times \beta^{(r)}(W^l_r)$ ▷ update the associated weight
      end if
    end for
  end for
  $J = \min(J_{\max}, j)$
  $\text{idx} = \text{sort}\left(\beta^{1:j}_{(r)}\right)$
  $\mathcal{S}_{(r)} = \left\{P^{\text{idx}(m)}_{(r)}\right\}_{m=1}^{J}$ ▷ retain partial covers with highest weights
end for

Figure 7: Pseudocode for finding likely partial covers

7 Simulation results

We compare Gaussian mixture implementations of the different multisensor PHD filters in this report. Specifically the iterated-corrector PHD filter [2] and the general multisensor PHD filter [2] are considered. The iterated-corrector PHD filter sequentially processes information from each sensor and the final result depends on the order in which sensors are processed.

The target tracks are simulated using the constant velocity model. The targets are moving inside a $2000m \times 2000m$ square region for 100 time steps. Two targets are present initially and the third target arrives at time step $k = 66$ and stays till the end. Three sensors (s = 3) are used to make observations about the targets position (x and y coordinates) within the monitoring region. The sensor noise is additive Gaussian with zero mean and covariance matrix $\sigma_r^2 I$ with $\sigma_r^2 = 100m^2$. The clutter process for each sensor is Poisson with rate $\lambda = 5$. Two of the sensors have a fixed probability of detection $P_d = 0.95$. The probability of detection of the third sensor is changed gradually from 0.5 to 0.95. Two cases of sensor ordering are considered as in the paper [9]. In Case 1 the variable sensor is processed towards the end while in Case 2 the variable sensor is processed first.

The analysis methodology is similar to the work presented in [9]. The Gaussian mixture implementations of the different filters are based on the paper [10]. Five filter outputs are considered as follows: the iterated-corrector PHD filter (IC PHD) and the iterated-corrector PHD filter (IC CPHD) for Case 1 and Case 2, and the general multisensor PHD filter for Case 1. For the general multisensor PHD filter we use $P_{\max} = 2^{s+1} = 16$. The average OSPA error (average calculated over 10 Monte Carlo simulations) performance is shown in the figure 8 for each of these cases with different value of P_d for the variable sensor. Figure 8 can be compared with Figure 4 in [9]. The trends for the iterated-corrector PHD filter in both Case 1 and Case 2 match the trends reported in [9]. The general multisensor PHD filter performs better than the iterated-corrector PHD filter for both cases. The IC CPHD filter has very little dependence on the sensor ordering and their average error values almost overlap for the two cases. The IC CPHD filter is slightly better than the current general multisensor PHD filter implementation.

7.1 Discussion of results

We can analyse the performance of iterated-corrector PHD and CPHD filters for the following simplifying scenario. Assume that the clutter is negligible and the probabilities of detection are constant. In the iterated-corrector filters, the output obtained after processing the previous sensor is used as predicted density for the next sensor in a sequential manner. For the PHD filter if $D^{[j-1]}_{k+1|k+1}(\mathbf{x})$ and $D^{[j]}_{k+1|k+1}(\mathbf{x})$ are the PHD's obtained after processing the $(j-1)^{th}$ and $(j)^{th}$ sensor, then the update equation is given as follows

$$D^{[j]}_{k+1|k+1}(\mathbf{x}) = L_j(\mathbf{x})\, D^{[j-1]}_{k+1|k+1}(\mathbf{x}) \tag{101}$$

$$\text{where, } L_j(\mathbf{x}) = (1 - p_{d,j}) + \sum_{\mathbf{z}\in Z_{k+1,j}} \frac{p_{d,j}\, h_j(\mathbf{z}|\mathbf{x})}{D^{[j-1]}_{k+1|k+1}[p_{d,j}\, h_j(\mathbf{z})]} \tag{102}$$

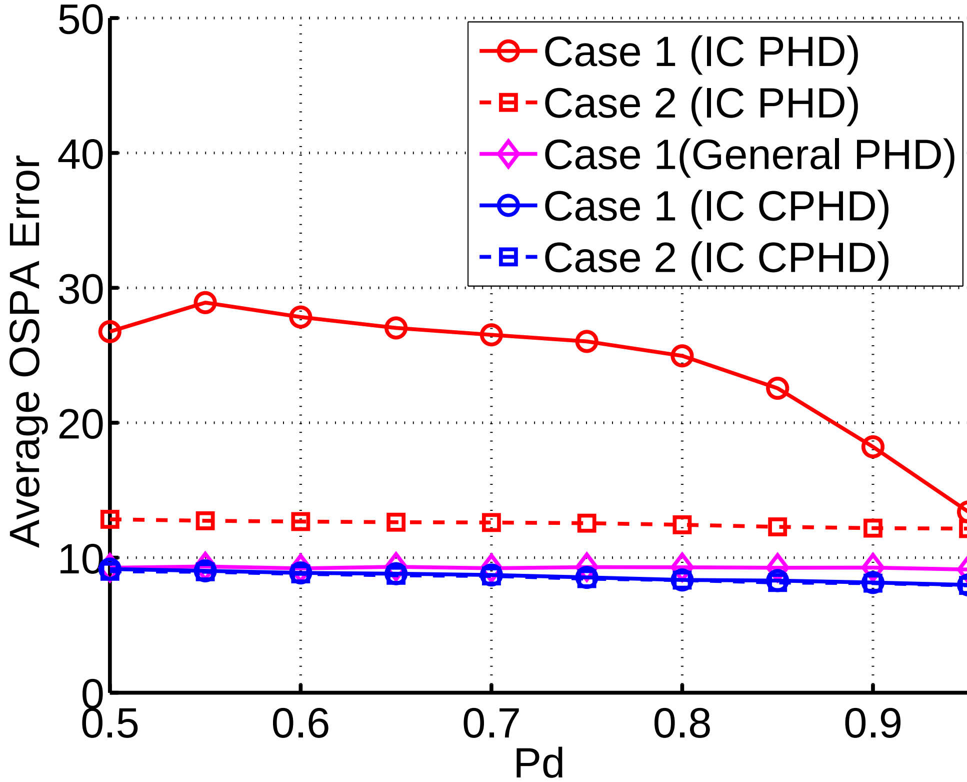

Figure 8: Average OSPA error vs P_d

Consider the case with 3 sensors where the first two sensors have high probability of detection $p_{d,1} \approx 1, p_{d,2} \approx 1$ and the third sensor has low probability of detection $p_{d,3}$. If the sensors with high probability of detections are processed first, $D^{[2]}_{k+1|k+1}(\mathbf{x})$ provides an accurate description of the targets. When the last sensor is processed, since it has low probability of detection, Gaussian componenets which represent the targets which do not get detected can only get a maximum weight of $(1 - p_{d,3})$. This happens even if the previous sensor processing assigned them weights close to unity. The estimate of the expected number of targets, which is equal to the sum of weights of all the Gaussian components, can get significantly reduced in this situation. Thus giving high error for IC PHD in Case 1 in the above simulations. When the sensor processing order is reversed, the subsequent detections by high probability of detection sensors corrects the error made by the first sensor processing.

The effect of sensor order is much reduced for the iterated-corrector CPHD filter because of additional propagation of the cardinality information. Let $p^{[j-1]}_{k+1|k+1}(n)$ and $p^{[j]}_{k+1|k+1}(n)$ be the cardinality distributions obtained after processing the $(j-1)^{th}$ and $(j)^{th}$ sensor. Then we have the following relation,

$$p^{[j]}_{k+1|k+1}(n) \propto p^{[j-1]}_{k+1|k+1}(n) \sum_{i=0}^{\min(n,m_j)} \frac{n!}{(n-i)!} C^{(m_j-i)}_{k+1,j}(0)\,(1-p_{d,j})^{n-i} \sigma_i(Z_{k+1,j}) \tag{103}$$

where the proportionality is up to a normalizing constant. When the clutter is negligible we have $C_{k+1,j}(t) = 1$ and the expression for $\sigma_i(Z_{k+1,j})$ is given in Section 2.1. Thus we have,

$$p^{[j]}_{k+1|k+1}(n) \propto \begin{cases} \approx 0 & \text{if } n < m_j \\ p^{[j-1]}_{k+1|k+1}(n) \frac{n!}{(n-m_j)!}\,(1-p_{d,j})^{n-m_j} \sigma_{m_j}(Z_{k+1,j}) & \text{if } n \geq m_j \end{cases} \tag{104}$$

When high probability of detection sensors are processed first, after processing two sensors the cardinality distribution $p^{[2]}_{k+1|k+1}(n)$ is approximately given by

$$p^{[2]}_{k+1|k+1}(n) = \begin{cases} \approx 0 & \text{if } n \neq N_{k+1} \\ \approx 1 & \text{if } n = N_{k+1} \end{cases} \tag{105}$$

where N_{k+1} is the actual number of targets present. When the last sensor is processed, even if we have misdetections, i.e. $m_3 < N_{k+1}$, combining equations (104) and (105) we have,

$$p^{[3]}_{k+1|k+1}(n) \propto \begin{cases} \approx 0 & \text{if } n \neq N_{k+1} \\ \approx \dfrac{N_{k+1}!}{(N_{k+1}-m_3)!}\,(1-p_{d,3})^{N_{k+1}-m_3} \sigma_{m_3}(Z_{k+1,3}) & \text{if } n = N_{k+1} \end{cases} \tag{106}$$

After normalization we get $p^{[3]}_{k+1|k+1}(n) = 1$ at $n = N_{k+1}$ and 0 otherwise. Thus when the updated cardinality distribution is used for estimating the number of targets (ex. using the MAP estimator), we still get correct cardinality. Note that the above analysis is valid when $p_{d,1} \approx 1, p_{d,2} \approx 1$. If this is not true, i.e. when the initially processed sensors do not have high probability of detection, then the IC CPHD filter can show significant dependence on the sensor order. In our simulations we have $p_{d,1} = p_{d,2} = 0.95$ hence we see very little dependence on sensor order for the IC CPHD filter for Case 1 and Case 2.

A Expression for collection of partitions

Define $Z^{(r)} = \bigcup_{j=1}^{r} Z^j$ and let $\mathcal{S}^{(r)}$ be the collection of all r-ary partitons of all possible subsets of $Z^{(r)}$. Let $Z^{r+1} = \{\mathbf{z}_1^{r+1}, \mathbf{z}_2^{r+1}, \ldots, \mathbf{z}_{m_{r+1}}^{r+1}\}$. We can express $\mathcal{S}^{(r+1)}$ using $\mathcal{S}^{(r)}$ and Z^{r+1} as given by the following relation

$$\mathcal{S}^{(r+1)} = \bigcup_{P \in \mathcal{S}^{(r)}} \bigcup_{n_1=0}^{m_{r+1}} \bigcup_{n_2=0}^{\min(m_{r+1},|P|)} \bigcup_{\substack{I_1 \subseteq [1\ m_{r+1}] \\ |I_1|=n_1}} \bigcup_{\substack{I_2 \subseteq [1\ m_{r+1}] \\ |I_2|=n_2 \\ I_1 \cap I_2 = \phi}} \bigcup_{\substack{J \subseteq [1\ |P|] \\ |J|=|I_2|}} \bigcup_{\sigma \in B(I_2,J)} \Big\{ \{W_j\}_{j \notin J} \cup \{\mathbf{z}_{i_1}^{r+1}\}_{i_1 \in I_1} \cup \{W_{\sigma(i_2)}, \mathbf{z}_{i_2}^{r+1}\}_{i_2 \in I_2} \Big\}$$

where $B(I_2, J)$ is the collection of all possible bijections from set I_2 to set J. (Proof required ?? or illustrate with an example ??).

As a special case,

$$\mathcal{S}^{(1)} = \bigcup_{n=0}^{m_1} \bigcup_{\substack{I \subseteq [1\ m_1] \\ |I|=n}} \Big\{ \{\mathbf{z}_i^{(1)}\}_{i \in I} \Big\} \tag{107}$$

B Proof of set derivatives with respect to Z

We have, for the case of s sensors

$$F[g_1, g_2, \ldots, g_s, h] = \left(\prod_{j=1}^{s} C_j(c_j[g_j]) \right) G(s[h \prod_{j=1}^{s} \phi_{g_j}]) \tag{108}$$

Let Z^j be the observation set due to sensor j, $j = 1, 2, \ldots, s$. Define

$$Z^{(r)} = \bigcup_{j=1}^{r} Z^j \tag{109}$$

$$\mathcal{S}^{(r)} = \text{collection of all } r-\text{ary partitons of all possible subsets of } Z^{(r)} \tag{110}$$

$$\mathcal{S} = \mathcal{S}^{(s)} \tag{111}$$

The derivation is based on the approach used by Delande et al. in their report [3] for deriving the general multisensor PHD filter. We prove using mathematical induction on $1 \le r \le s$ the following result,

$$\frac{\delta F}{\delta Z^1 \delta Z^2 \ldots \delta Z^r}[g_1, g_2, \ldots, g_s, h] = \Gamma^{(r)} \sum_{P \in \mathcal{S}^{(r)}} \psi_P^{(r)}[g_1, g_2, \ldots, g_s, h] \prod_{W \in P} d_W[g_1, g_2, \ldots, g_s, h] \tag{112}$$

$$\text{where, } \Gamma^{(r)} = \prod_{j=1}^{r} \left(\prod_{\mathbf{z} \in Z^j} c_j(\mathbf{z}) \right) \tag{113}$$

$$\psi_P^{(r)}[g_1, g_2, \ldots, g_s, h] = \left(\prod_{j=1}^{r} C_j^{(m_j - |P|_j)}(c_j[g_j]) \right) \left(\prod_{j=r+1}^{s} C_j(c_j[g_j]) \right) G^{(|P|)}(s[h \prod_{j=1}^{s} \phi_{g_j}]) \tag{114}$$

$$|P| = \text{cardinality of the } r\text{-ary partition P} \tag{115}$$

$$|P|_j = \text{number of elements of P containing observation from sensor } j \tag{116}$$

$$\text{For } 1 \le t_1 < t_2 < \cdots < t_M \le r, \text{ let } W = \{\mathbf{z}^{t_1}, \mathbf{z}^{t_2}, \ldots, \mathbf{z}^{t_M}\} \tag{117}$$

$$\text{where, } \mathbf{z}^{t_i} \in Z^{t_i} \text{ and } |W| = M \in [1\ r], \quad \text{then} \tag{118}$$

$$d_W[g_1, g_2, \ldots, g_s, h] = \frac{s\left[h \left(\prod_{i=1}^{M} p_d^{t_i} h_{t_i}(\mathbf{z}^{t_i}) \right) \left(\prod_{j \notin \{t_1, \ldots, t_M\}} \phi_{g_j} \right) \right]}{\prod_{i=1}^{M} c_{t_i}(\mathbf{z}^{t_i})} \tag{119}$$

We first establish the result for $r = 1$.

$$\begin{aligned}
\frac{\delta F}{\delta Z^1}[g_1, g_2, \ldots, g_s, h] &= \frac{\delta}{\delta Z^1} \left\{ \left(\prod_{j=1}^{s} C_j(c_j[g_j]) \right) G(s[h \prod_{j=1}^{s} \phi_{g_j}]) \right\} \\
&= \left(\prod_{j=2}^{s} C_j(c_j[g_j]) \right) \frac{\delta}{\delta Z^1} \left\{ C_1(c_1[g_1]) \, G(s[h \prod_{j=1}^{s} \phi_{g_j}]) \right\} \\
&= \left(\prod_{j=2}^{s} C_j(c_j[g_j]) \right) \left\{ \sum_{n=0}^{m_1} \sum_{\substack{I \subseteq [1\ m_1] \\ |I| = n}} \frac{\delta}{\delta\{\mathbf{z}_i^1\}_{i \in I}} G(s[h \prod_{j=1}^{s} \phi_{g_j}]) \frac{\delta}{\delta\{\mathbf{z}_i^1\}_{i \notin I}} C_1(c_1[g_1]) \right\} \\
&= RHS
\end{aligned}$$

We can show that,

$$\frac{\delta}{\delta\{\mathbf{z}_i^1\}_{i \in I}} G(s[h \prod_{j=1}^{s} \phi_{g_j}]) = G^{(n)}(s[h \prod_{j=1}^{s} \phi_{g_j}]) \prod_{i \in I} s[h \, p_d^1 \, h_1(\mathbf{z}_i^1) \prod_{j=2}^{s} \phi_{g_j}] \tag{120}$$

$$\frac{\delta}{\delta\{\mathbf{z}_i^1\}_{i \notin I}} C_1(c_1[g_1]) = C_1^{(m_1 - n)}(c_1[g_1]) \prod_{i \notin I} c_1(\mathbf{z}_i^1) = C_1^{(m_1 - n)}(c_1[g_1]) \frac{\Gamma^{(1)}}{\prod_{i \in I} c_1(\mathbf{z}_i^1)} \tag{121}$$

$$RHS = \Gamma^{(1)} \left(\prod_{j=2}^{s} C_j(c_j[g_j]) \right) \tag{122}$$

$$\sum_{n=0}^{m_1} \sum_{\substack{I \subseteq [1\ m_1] \\ |I| = n}} G^{(n)}(s[h \prod_{j=1}^{s} \phi_{g_j}]) \, C_1^{(m_1 - n)}(c_1[g_1]) \prod_{i \in I} \frac{s[h \, p_d^1 \, h_1(\mathbf{z}_i^1) \prod_{j=2}^{s} \phi_{g_j}]}{c_1(\mathbf{z}_i^1)} \tag{123}$$

Using result of Appendix A

$$RHS = \Gamma^{(1)} \sum_{P\in\mathcal{S}^{(1)}} C_1^{(m_1-|P|)}(c_1[g_1]) \left(\prod_{j=2}^{s} C_j(c_j[g_j])\right) G^{(|P|)}(s[h\prod_{j=1}^{s}\phi_{g_j}]) \prod_{W\in P} d_W \tag{124}$$

$$= \Gamma^{(1)} \sum_{P\in\mathcal{S}^{(1)}} \psi_P^{(1)}[g_1,g_2,\dots,g_s,h] \prod_{W\in P} d_W[g_1,g_2,\dots,g_s,h] \tag{125}$$

Hence the result is establised for the case $r=1$.

Now assuming that the result is true for some $r=b\geq 1$, we establish that the result holds for $r=b+1$. Let $Z^{b+1}=\{\mathbf{z}_1^{b+1},\mathbf{z}_2^{b+1},\dots,\mathbf{z}_{m_{b+1}}^{b+1}\}$.

$$\frac{\delta F}{\delta Z^1\delta Z^2\dots\delta Z^b\delta Z^{b+1}}[g_1,g_2,\dots,g_s,h] = \frac{\delta}{\delta Z^{b+1}}\left\{\frac{\delta F}{\delta Z^1\delta Z^2\dots\delta Z^b}[g_1,g_2,\dots,g_s,h]\right\} \tag{126}$$

$$= \frac{\delta}{\delta Z^{b+1}}\left\{\Gamma^{(b)}\sum_{P\in\mathcal{S}^{(b)}}\psi_P^{(b)}[g_1,g_2,\dots,g_s,h]\prod_{W\in P} d_W[g_1,g_2,\dots,g_s,h]\right\} \tag{127}$$

$$= \Gamma^{(b)}\left(\prod_{j=b+2}^{s} C_j(c_j[g_j])\right)\sum_{P\in\mathcal{S}^{(b)}}\left(\prod_{j=1}^{b} C_j^{(m_j-|P|_j)}(c_j[g_j])\right) \tag{128}$$

$$\times\frac{\delta}{\delta Z^{b+1}}\left\{C_{b+1}(c_{b+1}[g_{b+1}])G^{(|P|)}(s[h\prod_{j=1}^{s}\phi_{g_j}])\prod_{W\in P} d_W[g_1,g_2,\dots,g_s,h]\right\} \tag{129}$$

$$= RHS \tag{130}$$

Consider,

$$\frac{\delta}{\delta Z^{b+1}}\left\{C_{b+1}(c_{b+1}[g_{b+1}])G^{(|P|)}(s[h\prod_{j=1}^{s}\phi_{g_j}])\prod_{W\in P} d_W[g_1,g_2,\dots,g_s,h]\right\} \tag{131}$$

$$= \sum_{n_1=0}^{m_{b+1}}\sum_{n_2=0}^{\min(m_{b+1},|P|)}\sum_{\substack{I_1\subseteq[1\ m_{b+1}]\\|I_1|=n_1}}\sum_{\substack{I_2\subseteq[1\ m_{b+1}]\\|I_2|=n_2;\ I_1\cap I_2=\phi}}\frac{\delta}{\delta\{\mathbf{z}_{i_1}^{b+1}\}_{i_1\in I_1}}\left(G^{(|P|)}(s[h\prod_{n=1}^{s}\phi_{g_n}])\right) \tag{132}$$

$$\frac{\delta}{\delta\{\mathbf{z}_{i_2}^{b+1}\}_{i_2\in I_2}}\left(\prod_{W\in P} d_W[g_1,g_2,\dots,g_s,h]\right)\frac{\delta}{\delta\{\mathbf{z}_{i}^{b+1}\}_{i\notin I_1\cup I_2}}\left(C_{b+1}(c_{b+1}[g_{b+1}])\right) \tag{133}$$

Now

$$\frac{\delta}{\delta\{\mathbf{z}_{i_1}^{b+1}\}_{i_1\in I_1}}\left(G^{(|P|)}(s[h\prod_{n=1}^{s}\phi_{g_n}])\right) = G^{(|P|+n_1)}(s[h\prod_{n=1}^{s}\phi_{g_n}])\left(\prod_{i_1\in I_1} d_{\{\mathbf{z}_{i_1}^{b+1}\}}\, c_{b+1}(\mathbf{z}_{i_1}^{b+1})\right) \tag{134}$$

$$\frac{\delta}{\delta\{\mathbf{z}_{i_2}^{b+1}\}_{i_2\in I_2}}\left(\prod_{W\in P} d_W[g_1,g_2,\ldots,g_s,h]\right) \tag{135}$$

$$= \sum_{\substack{J\subseteq[1\ |P|]\\ |J|=|I_2|}} \sum_{\sigma\in B(I_2,J)} \left(\prod_{j\notin J} d_{W_j}[g_1,g_2,\ldots,g_s,h]\right)\left(\prod_{i_2\in I_2} d_{W_{\sigma(i_2)}\cup \mathbf{z}_{i_2}^{b+1}}[g_1,g_2,\ldots,g_s,h]c_{b+1}(\mathbf{z}_{i_2}^{b+1})\right) \tag{136}$$

where $B(I_2,J)$ is the collection of all possible bijections from set I_2 to set J.

$$\frac{\delta}{\delta\{\mathbf{z}_i^{b+1}\}_{i\notin I_1\cup I_2}}\left(C_{b+1}(c_{b+1}[g_{b+1}])\right) = C_{b+1}^{(m_{b+1}-n_1-n_2)}(c_{b+1}[g_{b+1}]) \prod_{i\notin I_1\cup I_2} c_{b+1}(\mathbf{z}_i^{b+1}) \tag{137}$$

Hence we have,

$$\begin{aligned} RHS = \Gamma^{(b)} &\left(\prod_{j=b+2}^{s} C_j(c_j[g_j])\right)\left(\prod_{\mathbf{z}^{b+1}\in Z^{b+1}} c_{b+1}(\mathbf{z}^{b+1})\right) \\ &\sum_{P\in\mathcal{S}^{(b)}} \sum_{n_1=0}^{m_{b+1}} \sum_{n_2=0}^{\min(m_{b+1},|P|)} \sum_{\substack{I_1\subseteq[1\ m_{b+1}]\\ |I_1|=n_1}} \sum_{\substack{I_2\subseteq[1\ m_{b+1}]\\ |I_2|=n_2;\ I_1\cap I_2=\phi}} \sum_{\substack{J\subseteq[1\ |P|]\\ |J|=|I_2|}} \sum_{\sigma\in B(I_2,J)} \\ &G^{(|P|+n_1)}(s[h\prod_{n=1}^{s}\phi_{g_n}])\, C_{b+1}^{(m_{b+1}-n_1-n_2)}(c_{b+1}[g_{b+1}]) \left(\prod_{j=1}^{b} C_j^{(m_j-|P|_j)}(c_j[g_j])\right) \\ &\left(\prod_{i_1\in I_1} d_{\{\mathbf{z}_{i_1}^{b+1}\}}\right)\left(\prod_{i_2\in I_2} d_{W_{\sigma(i_2)}\cup\mathbf{z}_{i_2}^{b+1}}[g_1,g_2,\ldots,g_s,h]\right)\left(\prod_{j\notin J} d_{W_j}[g_1,g_2,\ldots,g_s,h]\right) \end{aligned}$$

Using result of Appendix A we have

$$\begin{aligned} RHS &= \Gamma^{(b+1)} \sum_{P\in\mathcal{S}^{(b+1)}} \left(\prod_{j=1}^{b+1} C_j^{(m_j-|P|_j)}(c_j[g_j])\right)\left(\prod_{j=b+2}^{s} C_j(c_j[g_j])\right) \\ &\quad G^{(|P|)}(s[h\prod_{n=1}^{s}\phi_{g_n}])\left(\prod_{W\in P} d_W[g_1,g_2,\ldots,g_s,h]\right) \\ &= \Gamma^{(b+1)} \sum_{P\in\mathcal{S}^{(b+1)}} \psi_P^{(b+1)}[g_1,g_2,\ldots,g_s,h] \prod_{W\in P} d_W[g_1,g_2,\ldots,g_s,h] \end{aligned}$$

C Proof of set derivative with respect to x

For brevity denote $d_W[0,0,\dots,0,h] = d_W[h]$.

$$\frac{\delta F}{\delta \mathbf{x} \delta Z^1 \delta Z^2 \dots \delta Z^s}[0,0,\dots,0,h] = \frac{\delta}{\delta \mathbf{x}} \left\{ \frac{\delta F}{\delta Z^1 \delta Z^2 \dots \delta Z^s}[0,0,\dots,0,h] \right\} \tag{138}$$

$$= \frac{\delta}{\delta \mathbf{x}} \left\{ \Gamma \sum_{P \in \mathcal{S}} \psi_P[0,0,\dots,0,h] \prod_{W \in P} d_W[h] \right\} \tag{139}$$

$$= \Gamma \sum_{P \in \mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) \frac{\delta}{\delta \mathbf{x}} \left\{ G^{(|P|)}(s[h \prod_{j=1}^{s} q_d^j]) \prod_{W \in P} d_W[h] \right\} \tag{140}$$

$$= \Gamma \sum_{P \in \mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|+1)}(s[h \prod_{j=1}^{s} q_d^j]) \left(s(\mathbf{x}) \prod_{j=1}^{s} q_d^j \right) \prod_{W \in P} d_W[h] \tag{141}$$

$$+ \Gamma \sum_{P \in \mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|)}(s[h \prod_{j=1}^{s} q_d^j]) \frac{\delta}{\delta \mathbf{x}} \prod_{W \in P} d_W[h] \tag{142}$$

Substituting $h \equiv 1$ we have

$$\frac{\delta F}{\delta \mathbf{x} \delta Z^1 \delta Z^2 \dots \delta Z^s}[0,0,\dots,0,1] \tag{143}$$

$$= \Gamma \sum_{P \in \mathcal{S}} \left\{ \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|+1)}(s[\prod_{j=1}^{s} q_d^j]) \right\} \prod_{W \in P} d_W \left(s(\mathbf{x}) \prod_{j=1}^{s} q_d^j \right) \tag{144}$$

$$+ \Gamma \sum_{P \in \mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|)}(s[\prod_{j=1}^{s} q_d^j]) \left(\sum_{W \in P} s(\mathbf{x}) \, \rho_W(\mathbf{x}) \right) \prod_{W \in P} d_W \tag{145}$$

$$= \Gamma \sum_{P \in \mathcal{S}} \left(\psi_P^* \prod_{W \in P} d_W \right) \times \left(s(\mathbf{x}) \prod_{j=1}^{s} q_d^j \right) \tag{146}$$

$$+ \Gamma \sum_{P \in \mathcal{S}} \left\{ \left(\prod_{j=1}^{s} C_j^{(m_j - |P|_j)}(0) \right) G^{(|P|)}(s[\prod_{j=1}^{s} q_d^j]) \right\} \prod_{W \in P} d_W \left(\sum_{W \in P} s(\mathbf{x}) \, \rho_W(\mathbf{x}) \right) \tag{147}$$

$$= \Gamma \sum_{P \in \mathcal{S}} \left(\psi_P^* \prod_{W \in P} d_W \right) \times \left(s(\mathbf{x}) \prod_{j=1}^{s} q_d^j \right) + \Gamma \sum_{P \in \mathcal{S}} \left(\psi_P \prod_{W \in P} d_W \right) \times \left(s(\mathbf{x}) \sum_{W \in P} \rho_W(\mathbf{x}) \right) \tag{148}$$

D Derivation of cardinality distribution

The PGF of the posterior cardinality distribution is given by

$$G_{k+1|k+1}(t) = G_{k+1|k+1}[t] = \frac{\sum_{P\in\mathcal{S}} \psi_P[0,0,\ldots,0,t] \prod_{W\in P} d_W[0,0,\ldots,0,t]}{\sum_{P\in\mathcal{S}} \psi_P \prod_{W\in P} d_W} \tag{149}$$

We have $\prod_{W\in P} d_W[0,0,\ldots,0,t] = t^{|P|} \prod_{W\in P} d_W$. Define $\gamma = s[\prod_{j=1}^{s} q_d^j]$. By definition

$$\begin{aligned}
p_{k+1|k+1}(n) &= \frac{1}{n!} G^{(n)}_{k+1|k+1}(0) && (150)\\
&= \frac{1}{n!}\left\{\frac{d^n}{dt^n}\frac{\sum_{P\in\mathcal{S}} t^{|P|}\psi_P[0,0,\ldots,0,t] \prod_{W\in P} d_W}{\sum_{P\in\mathcal{S}} \psi_P \prod_{W\in P} d_W}\right\}_{t=0} && (151)\\
&= \left\{\frac{\sum_{P\in\mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right) \frac{1}{n!}\frac{d^n}{dt^n}\left\{t^{|P|} G^{(|P|)}(t\gamma)\right\} \prod_{W\in P} d_W}{\sum_{P\in\mathcal{S}} \psi_P \prod_{W\in P} d_W}\right\}_{t=0} && (152)
\end{aligned}$$

We have the following result

$$\frac{1}{n!}\left\{\frac{d^n}{dt^n} t^{|P|} G^{(|P|)}(t\gamma)\right\}_{t=0} = \begin{cases} 0 & \text{if } n < |P| \\ \dfrac{1}{(n-|P|)!} G^{(n)}(0)\,\gamma^{n-|P|} & \text{if } n \geq |P| \end{cases} \tag{153}$$

Hence,

$$p_{k+1|k+1}(n) = \frac{\sum_{\substack{P\in\mathcal{S}\\ |P|\leq n}} \frac{1}{(n-|P|)!}\left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right) G^{(n)}(0)\,\gamma^{n-|P|} \prod_{W\in P} d_W}{\sum_{P\in\mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right) G^{(|P|)}(\gamma) \prod_{W\in P} d_W} \tag{154}$$

Since $G^{(n)}(0) = n!\, p_{k+1|k}(n)$,

$$\frac{p_{k+1|k+1}(n)}{p_{k+1|k}(n)} = \frac{\sum_{\substack{P\in\mathcal{S}\\ |P|\leq n}} \frac{n!}{(n-|P|)!}\left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right) \gamma^{n-|P|} \prod_{W\in P} d_W}{\sum_{P\in\mathcal{S}} \left(\prod_{j=1}^{s} C_j^{(m_j-|P|_j)}(0)\right) G^{(|P|)}(\gamma) \prod_{W\in P} d_W} \tag{155}$$

E Generalized partitions

References

[1] C. Ouyang and H. Ji, “Scale unbalance problem in product multisensor PHD filter,” *Electronics letters*, vol. 47, no. 22, pp. 1247–1249, 2011.

[2] R. Mahler, “The multisensor PHD filter: I. General solution via multitarget calculus,” in *Proc. SPIE Int. Conf. Sig. Proc., Sensor Fusion, Target Recog.*, Orlando, FL, U.S.A., Apr. 2009.

[3] E. Delande, E. Duflos, D. Heurguier, and P. Vanheeghe, “Multi-target PHD filtering: proposition of extensions to the multi-sensor case,” 2010.

[4] R. Mahler, “Multitarget Bayes filtering via first-order multitarget moments,” *IEEE Trans. Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1152–1178, Oct. 2003.

[5] J. Xu, F.-M. Huang, and Z.-L. Huang, “The multi-sensor PHD filter: Analytic implementation via Gaussian mixture and effective binary partition,” in *Proc. Int. Conf. Inf. Fusion*, Jul. 2013, pp. 945–952.

[6] Exact cover problem. [Online]. Available: http://en.wikipedia.org/wiki/Exact_cover

[7] D. E. Knuth, “Dancing links,” *arXiv preprint cs/0011047*, 2000.

[8] A. O’Dwyer. Dancing links C code. [Online]. Available: http://www.club.cc.cmu.edu/~ajo/free-software/xword/dancing.c

[9] S. Nagappa and D. E. Clark, “On the ordering of the sensors in the iterated-corrector probability hypothesis density (PHD) filter,” in *Proc. SPIE Int. Conf. Sig. Proc., Sensor Fusion, Target Recog.*, Orlando, FL, U.S.A., Apr. 2011.

[10] B.-N. Vo and W.-K. Ma, “The Gaussian mixture probability hypothesis density filter,” *Sig. Proc., IEEE Tran.*, vol. 54, no. 11, pp. 4091–4104, 2006.