a probabilistic view of locality in graph signal processing

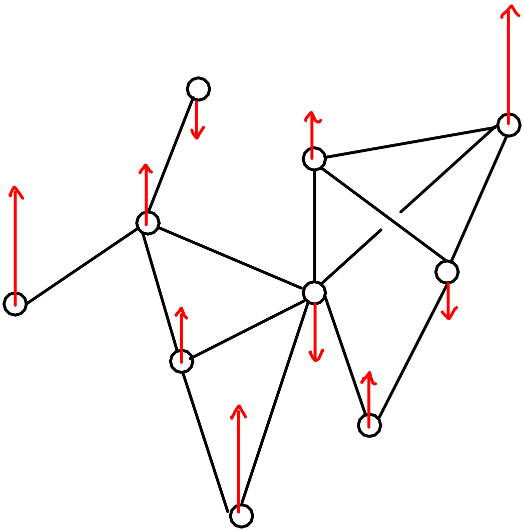
Mitch Roddenberry
Fernando Gama
Richard Baraniuk
Santiago Segarra

Rice ECE – Houston, R.o.T.

Bellairs Workshop – December 2021

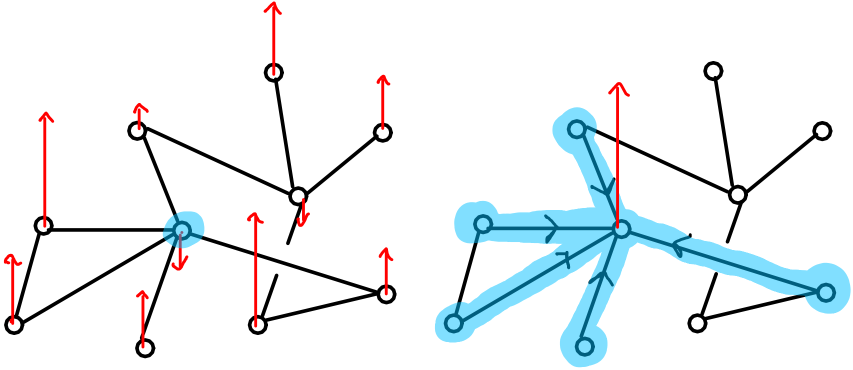
graphs and graph signals

- A finite graph $G = (V, E)$
- Functions on its nodes
  $\mathbb{X}(G) = \{x : V \to \mathbb{R}\}$



graph shift operators

Graph shift operators “diffuse” signals using the local topology



sparse matrix-vector multiplication

- For $x \in \mathbb{X}(G)$, $v \in V$, graph shift operators follow
$$[Sx]_v = \sum_{u \in N(v)} [S]_{vu}[x]_u$$
- Identify $\mathbb{X}(G)$ with $\mathbb{R}^n$
  - S becomes a square $n \times n$ matrix
  - x becomes a vector in $\mathbb{R}^n$

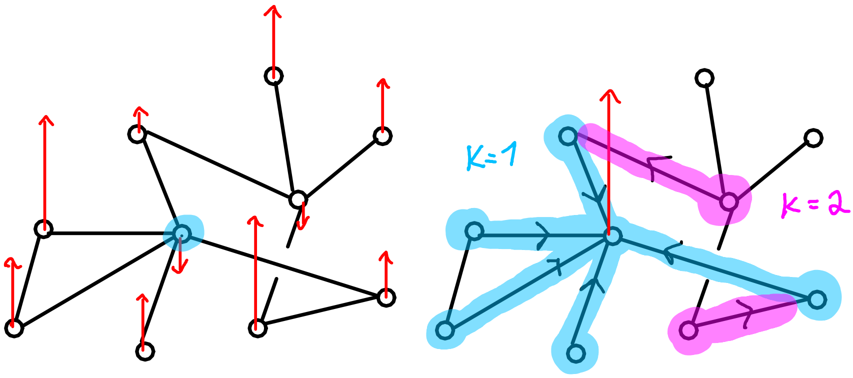
graph filtering

- Graph shift operators are "one-hop" diffusions
- A *graph filter of degree K* is simply a degree K polynomial:

$$H(S) = \sum_{k=0}^{K} h_k S^k$$

- Yields the following *locality property*:

$[H(S)x]_v$ only depends on the signal and topology of $N^K(v)$


k=1
k=2

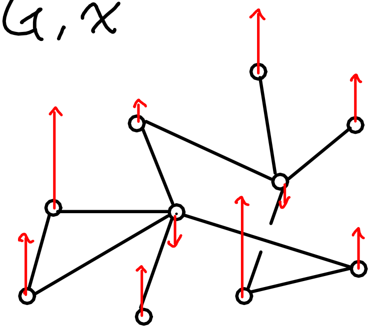
key questions

- What invariants are important in graph filtering?
- How to compare behavior of one filter across two graphs?
- Spectral analysis?
- How can graph signals be understood in the limit?

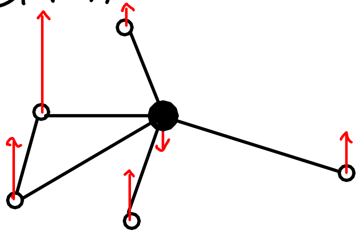
machinery: rooted balls

- A *rooted graph* is a graph with a root
  - If a graph is a tuple $G = (V, E)$,
  - A rooted graph is a triple $\bar{G} = (V, E, r)$, for some $r \in V$
- Signals are the same: $\mathbb{X}(\bar{G}) = \{x : V \to \mathbb{R}\}$
- A *rooted K-ball* is a rooted graph *of radius K*
- Denote by $\bar{B}_K(v)$ the K-ball centered at v, for $v \in V$
- The corresponding signal by $\bar{x}_K(v)$

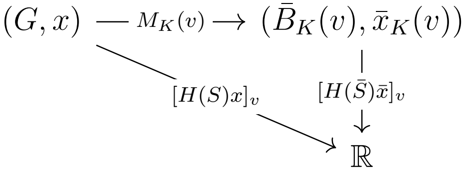
G, x



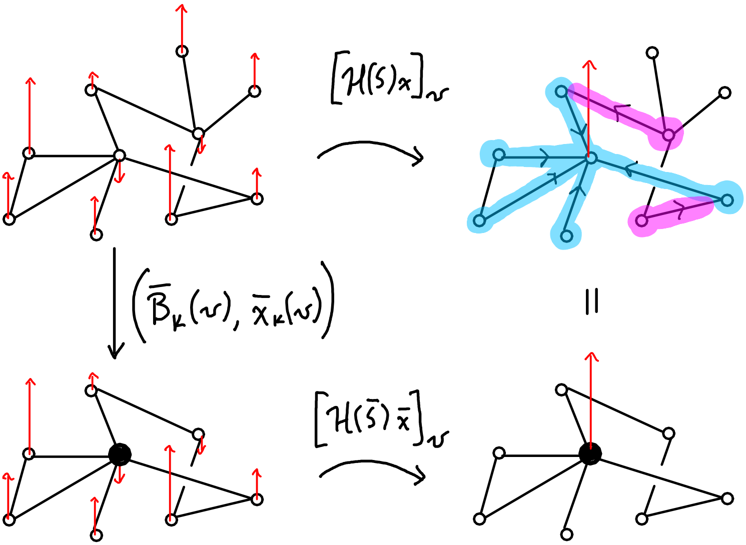
$\bar{B}_\kappa(v), \bar{x}_\kappa(v)$



- K-motifs are elements of
  $$\Omega_K = \coprod_{\bar{G}:\mathrm{rad}(\bar{G})\leq K} \mathbb{X}(\bar{G})$$
- Define $M_K : V \to \Omega_K$ as
  $$M_K(v) = (\bar{B}_K(v), \bar{x}_K(v))$$
- The diagram commutes



locality of graph filtering



probabilistic graph representations

- Ω_K is a regular, Hausdorff topological space
- *Approach*: a graph with a signal is just a big bag of motifs
- For a graph $G = (V, E)$ and signal $x \in \mathbb{X}(G)$, let U be the uniform probability measure on V
- Define μ as the pushforward of U by M_K

$$\mu = (M_K)_*(U)$$

consequences

- The probability measure μ on Ω_K does not care too much about the size of the underlying graph
- A means to look at graphs and graph signals in a way that does not depend on them having the same size
- Look at graphs through the lens of K-hop functions

spectral analysis of graph signals

- A GSO of special interest: the graph Laplacian

$$[\Delta]_{uv} = \begin{cases} \deg(v) & u = v \\ -1 & (u,v) \in E \\ 0 & \text{else.} \end{cases}$$

- Spectrum contained in $[0, 2 \cdot d_{\max}]$
- Measures signal smoothness in the following way

$$\langle x, \Delta x \rangle = \sum_{(u,v)\in E} (x(v) - x(u))^2$$

why are these called Fourier modes

- Let the eigenpairs of Δ be (λ_j, z_j) for $1 \leq j \leq |V|$
- $\lambda_j = \langle z_j, \Delta z_j \rangle$

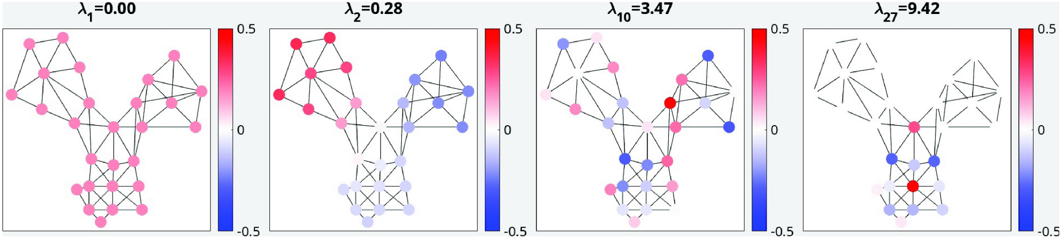
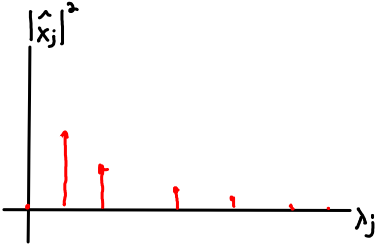
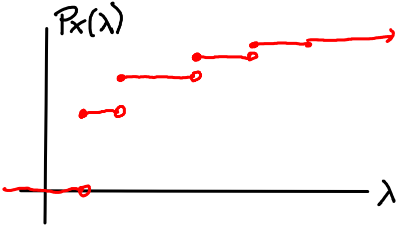


Figure from (Ortega *et. al.*, 2018)

the power spectral measure

- The eigenvectors z_j form an orthobasis for $\mathbb{X}(G)$
- The "graph Fourier transform" represents a signal in this basis
$$\hat{x}_j = \langle z_j, x \rangle$$
- Define a power distribution function $P_x : \mathbb{R} \to \mathbb{R}$
$$P_x(\lambda) = \frac{1}{|V|} \sum_{j:\lambda_j \leq \lambda} \hat{x}_j^2$$
- A finite measure on $[0, 2 \cdot d_{\max}]$


$|\hat{x}_j|^2$
λ_j


$P_x(\lambda)$
λ

moments of the power spectral measure

- For $x \in \mathbb{X}(G)$ with GFT $\hat{x}$, define

$$m_K(x) := \int_{\mathbb{R}} \lambda^K dP_x(\lambda) = \frac{1}{|V|} \langle x, \Delta^K x \rangle$$

- For $((V, E, r), x) \in \Omega_K$, put

$$\bar{m}_K((V, E, r), x) = [x]_r \cdot [\Delta^K x]_r$$

- It holds that

$$m_K(x) = \mathbb{E}_\mu[\bar{m}_K]$$

descencion to a local map

The following diagram commutes

$$\begin{array}{ccc} (G, x) & \xrightarrow{(M_K)_*} & \mu \\ & \overset{m_k}{\searrow} & \Big\downarrow {\scriptstyle \mathbb{E}[\bar{m}_K]} \\ & & \mathbb{R} \end{array}$$

why one ought to care

- For a $K/2$-tap graph filter $H(\Delta)$ with coefficients $\{h_k\}_{k=0}^{K}$
- The MSE under AWGN η is given by

$$\mathbb{E}\left[\frac{1}{|V|}\|x - H(\Delta)(x+\eta)\|_2^2\right] =$$

$$\int_{\mathbb{R}} \underbrace{(1 - H(\lambda))^2}_{\text{degree } K \text{ polynomial}} dP_x(\lambda) + \int_{\mathbb{R}} \underbrace{(H(\lambda))^2}_{\text{degree } K \text{ polynomial}} dP_\eta(\lambda)$$

- Performance in terms of integrals of power spectral measure
- If you know enough moments, you know the MSE

convergence

- Let $\{G_n, x_n\}_{n=1}^{\infty}$ be a sequence of graphs and graph signals satisfying the following assumptions:
    1. The nodes of the graphs have uniformly bounded degree ($d_{\max} = D$)
    2. The graph signals are uniformly bounded

Theorem

- *Let $K \geq 0$ be given*
- *Denote by μ_n the pushforward measure of (G_n, x_n)*
- *If the measures μ_n converge weakly, then $m_K(x_n)$ converges*
- *If this holds for all K, the measures P_{x_n} converge weakly*

proof sketch

Lemma

There exists a compact subspace $A \subseteq \Omega_K$ *such that for all bounded degree graphs with bounded signals, the measure* μ *satisfies* $\operatorname{supp}(\mu) \subseteq A$

- Compactness: all continuous functions are bounded
- $\bar{m}_K$ is continuous, thus bounded
- Weak convergence of measure implies convergence of expectations
- Weak convergence of power measures: Stone-Weierstrass theorem

finite approximation

Can we approximate arbitrarily large graphs with small graphs?

Theorem

- *Suppose a "graph signal property" J descends to the expectation of a continuous function $\bar{J}$ on Ω_K*
- *Let $\epsilon > 0$ be given*
- *There is an $n(\epsilon) < \infty$ such that for any (G, x) of degree D and signal in $[-1, 1]$, there exists a graph/signal (G_0, x_0) on at most $n(\epsilon)$ nodes where $|J(G, x) - J(G_0, x_0)| < \epsilon$*

proof sketch (1)

- Let $\Omega_{K,D}[-1,1]$ be the compact subspace of Ω_K that supports all graphs of degree bounded by D with signals contained in $[-1,1]$
- Ω_K is very nice $\to$ $\Omega_{K,D}[-1,1]$ admits a metric structure
  - Urysohn's metrization theorem

proof sketch (2)

- If $\bar{J}$ is continuous, it is (ϵ, δ)-uniformly continuous on $\Omega_{K,D}[-1, 1]$
- By Prokhorov's theorem, the set of probability measures of bounded graphs is compact
- Can argue for the continuity of J by descent to $\bar{J}$
- Typical maximal packing arguments for function approximation: put $n(\epsilon)$ to be the maximum graph size of a maximal $\delta/2$-packing of the space

further considerations

- Looking at motif distributions in graph signal processing can be used to understand the graph Fourier transform
  - This talk essentially looked at polynomials on $\mathbb{R}^n$
  - Attach any compact feature space to the nodes (GNNs)

- Compare graphs using integral probability metrics
  - Metrize Ω_K, yields a meaningful Wasserstein 1-distance between graphs based on motif densities via the pullback of the metric

- Theory of graph limits
  - *Graphons* and signals on them are studied by Ruiz, Chamon, Ribeiro, as well as Morency & Leus
  - Only handles dense graph limits: unbounded degree
  - Appropriate limit objects for bounded degree (very sparse) graphs: *graphings* (Lovász, 2012)